The 21st Seminar on Mathematical Analysis and its Applications

26-27 November 2014, Islamic Azad University, Hamedan, Iran, pp 367-369

Some results on Bishop-Phelps type theorem for s-cones

I. Sadeqi [1], A. Hassanzadeh*[2], A. Ranjbari[3]

[1] Academic member of Departement of Mathematics, Sahand University of Technology, esadeqi@sut.ac.ir

[2] Ph. D. student of Departement of Mathematics, Sahand University of Technology, a_hassanzadeh@sut.ac.ir

[3] Academic member of Departement of Pure Mathematics, University of Tabriz, ranjbari@tabrizu.ac.ir

Abstract

In this paper, the notion of support points of a set in a semitopological cone is introduced, and it's shown that any nonempty convex Scott-closed bounded set has a support point in a complete cancellative locally convex s-cone. Moreover, we prove the Bishop-Phelps type theorem for s-cones.

Keywords: Semitopological cone; S-cone; Scott topology; Support point; Bishop-Phelps theorem.

2010 MSC: 46N10; 47L07; 54D10.

1. Introduction and Preliminaries

Let S be a nonempty subset of a real Banach space X and ϕ be a nonzero continuous linear functional on X. If ϕ attains either its maximum or its minimum over S at the point $x \in S$, we say that ϕ supports S at x and that x is a support point of S.

In 1958 [4] Victor Klee asked if each closed bounded convex subset of a Banach space must have a support point. In 1961 E. Bishop and R.R. Phelps in their fundamental paper [2] answered affirmatively the Klee's question.

In 2000 [5] W. Roth has established the separation theorem in locally convex cones, and then in [6] R. Tix proved the some result in continuous d-cones, with dropping some conditions. Finally K. Keimel [3] improved the separation theorem for semi-topological cones.

Since the Bishop-Phelps theorem [2] plays an essential role in the proof of the separation theorem, so it is of interest to know whether Bishop-Phelps theorem is remains true in semi-topological cones.

For subsets A of a partially ordered set P we use the following notations:

$$\downarrow A =_{def} \{x \in P | x \leq a \ for \ some \ a \in A\},$$

$$\uparrow A =_{def} \{x \in P | x \geq a \; for \; some \; a \in A\}$$

and we say that A is a lower or upper set, if $\downarrow A = A$ or $\uparrow A = A$, respectively.

We denote by $\mathbb{R}_+$ the subset of nonnegative reals $r \geq 0$. Further $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ denote the reals and nonnegative reals extended by $+\infty$.

Any T_0-space X comes with an intrinsic order, the specialisation order which is defined by $x \leq y$ if the element x is contained in the closure of the singleton $\{y\}$.

Definition 1.1. A semitopological cone is a cone with a T_0-topology such that addition and scalar multiplication are separately continuous. (For the deffinition of a cone, see [3].)

Definition 1.2. An s-cone is a cone with a partial order such that addition and scalar multiplication: $(a, b) \mapsto a + b : C \times C \to C$, $(r, a) \mapsto ra : \mathbb{R}_+ \times C \to C$ are Scott-continuous.

Note that every s-cone is a semitopological cone with respect to its Scott topology.

Theorem 1.3. [3, Theorem 9.1] In a semitopological cone C consider a nonempty convex subset A and an open convex subset U. If A and U are disjoint, then there exists a Scott-continuous linear functional $\Lambda : C \to \overline{\mathbb{R}}_+$ such that $\Lambda(a) \leq 1 < \Lambda(u)$ for all $a \in A$ and all $u \in U$.

2. Main results

A cancellative cone (more precisely cancellative asymmetric cone) is a cone C, satisfying the following laws for all $v, w, u \in C$:

$$v + u = w + u \Rightarrow v = w \quad (cancellation),$$

$$v + w = 0 \Rightarrow v = w = 0 \quad (strictness).$$

Let B be a convex Scott-closed set in a semitopological cone C. A point $x \in B$ is called a support point for B, if there exits a linear Scott-continuous functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(x) = \sup f(B)$; such functional f is said support functional.

Let B be a nonempty convex Scott-closed set in a semitopological cone C with empty interior. For investigating the situation of support points, we use the subcone. First we define some concepts.

Definition 2.1. Let C be a semitopological cone and $d \in C$;

a. The net x_α is a d-Cauchy net, if for any $\epsilon > 0$ there exits α_0 such that for $\alpha > \beta \geq \alpha_0$ we have $x_\alpha \leq x_\beta + \epsilon d$ and $x_\beta \leq x_\alpha + \epsilon d$.

b. The directed net x_α is convergent to x, if $\sup x_\alpha = x$.

c. A semitopological cone C is called complete, if every r-Cauchy net ($r \in C$), is convergent.

In this paper an order on a semitopological cone will be the specialisation order $\leq$.

Let C be a cancellative semitopological cone and $f : C \to \overline{\mathbb{R}}_+$ be a Scott-continuous linear functional. For $0 < \delta < 1$ and $d \in C$, we define

$$K(f, \delta, d) = \{x \in C : f(x) < \infty \; and \; \delta x \leq f(x).d\}$$

The following order defines a partial order on C:

$$x \sqsubseteq y \Leftrightarrow y \in x + K.$$

Applying Definition 1.1 we have the following lemma:

Lemma 2.2. Let C be a cancellative semitopological cone. Then for $x, y \in C$ we have

$$x \sqsubseteq y \ (y \in x + K) \Rightarrow x \leq y \ (\textit{with specialisation order})$$

Lemma 2.3. Let f be a Scott-continuous linear functional on a complete cancellative s-cone C, and let $0 < \delta < 1$ and $d \in C$ be given. If B is a nonempty convex bounded Scott-closed subset of C, then for each $b \in B$ there exits some $m \in B$ satisfying $B \cap (m + K(f, \delta, d)) = \{m\}$ and $b \sqsubseteq m$.

Proof. It is sufficient to show that the partially ordered set $(B_b = \{y \in B : y \geq b\}, \sqsubseteq)$ has a maximal element. □

Lemma 2.4. Let f be a Scott-continuous linear functional on a complete cancellative s-cone C, and let $0 < \delta < 1$ and $d \in C$ be given. If B is a nonempty convex bounded Scott-closed subset of C and $m \in B$ such that $B \cap (m + K(f, \delta, d)) = \{m\}$ then $B \cap (\uparrow (m + K(f, \delta, d) \backslash \{0\}))^{\circ} = \varnothing$.

Proof. The element of m is a minimum element of $\uparrow m + K$ and so $m \notin (\uparrow m + (K \backslash \{0\}))^{\circ}$. It follows that $B \cap (\uparrow (m + K(f, \delta, d) \backslash \{0\}))^{\circ} = \varnothing$ □

Applying the separation theorem and Lemmas 2.3 , 2.4 we conclude the following main result:

Theorem 2.5. Let B be a nonempty convex d-bounded Scott-closed set for some $d \in C$ in a complete cancellative locally convex s-cone C. Then we have the following:

1. For each $x_0 \in B$ with the property that for $\lambda > 1$, $\lambda x \notin B$, and for $\epsilon > 0$, there exist a Scott-continuous linear functional $\Lambda : C \to \overline{\mathbb{R}}_+$ and $m \in B$ such that $\Lambda(m) = \sup \Lambda(B)$ and $x_0 \leq m \leq x_0 + \epsilon.d$.

2. For each Scott-continuous linear functional $f : C \to \overline{\mathbb{R}}_+$, there exists a support functional h such that $0 \leq h \leq f$.

References

1. Abramsky, S. and Jung, A. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, volume 3, Clarendon Press (1994) 1-168.
2. Bishop, E. and Phelps, R. R. The support functionals of a convex set, Convexity, Proc. Sympos. Pure Math Vol. 7, Amer. Math. Soc., Providence, R. I., (1963) 27-35.
3. Keimel, K. Topological Cones: Functional Analysis in a T_0-Setting, Semigroup Forum 77, (2008) 109-142.
4. Klee, V. Extremal structures of convex sets, Math. Z. 69, (1958) 98.
5. Roth, W. Hahn-Banach type theorems for locally convex cones, J. Aust. Math. Soc. 68(1), (2000) 104 -125.
6. Tix, R. Some results on Hahn-Banach type theorems for continuous d-cones, Theor. Comput. Sci. 264, (2001) 205-218.