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Determinant representations of sequences

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Abstract

In this talk we recall some recent results concerning (integer) matrices whose leading principal minors form well-known sequences such as Fibonacci, Lucas, Jacobsthal and Pell (sub)sequences. There are many different ways of constructing such matrices. Some of these matrices are constructed by homogeneous or non-homogeneous recurrence relations, and others are constructed by convolution of two sequences. Here, we will illustrate the idea of some methods by constructing some integer matrices of this kind.

Keywords: Determinant, generalized Pascal triangle, (Quasi) Toeplitz matrix, (Quasi) Pascal-like matrix, Fibonacci (Lucas, Jacobsthal and Pell) sequence.

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1 Introduction

Let $A = [A_{i,j}]_{i,j\geqslant 0}$ be an arbitrary infinite dimensional matrix. We denote by $d_n(A)$ the *nth leading principal minor* of A, which is defined as follows:

$$d_n(A) = \det[A_{i,j}]_{0\leqslant i,j\leqslant n}, \quad (n = 0,1,2,3,\ldots).$$

We put $D(A) := (d_n(A))_{n\geqslant 0}$. Two infinite dimensional matrices A and B are said to be *equimodular* (see [21]) if $D(A) = D(B)$. Given a sequence $\omega = (\omega_n)_{n\geqslant 0}$, a family $\{A_i |\ i \in I\}$ of equimodular matrices are said to be *ω-equimodular* if $D(A_i) = \omega$ for all $i \in I$. We will denote the family of

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ω-equimodular matrices by $\mathscr{A}_\omega$. The infinite dimensional matrices in $\mathscr{A}_\omega$ are said to be determinant representations of ω. Note that for *any* sequence $\omega = (\omega_n)_{n\geqslant 0}$, there is a determinant representation of ω, in other words $\mathscr{A}_\omega \neq \emptyset$. Indeed, expanding along the last rows, it is easy to see that

$$\begin{pmatrix} \omega_0 & 1 & * & * & * & \cdots \\ -\omega_1 & 0 & 1 & * & * & \cdots \\ \omega_2 & 0 & 0 & 1 & * & \cdots \\ -\omega_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathscr{A}_\omega,$$

(see also Theorem 3.2 and the Remark after this theorem in [20]).

Here, we will present several families of infinite dimensional (equimodular) matrices in which the sequence of their leading principal minors is a subsequence of the *Fibonacci, Lucas, Jacobsthal* or *Pell sequence*. It is worthwhile to point out that many of such matrices are *integer matrices* (recall that an integer matrix is a matrix whose entries are all integers).

2 Notation and Definitions

In this brief section, we introduce the basic definitions and notation that will be used throughout this note at various places. A *Gibonacci sequence* (generalized Fibonacci sequence) is the numerical sequence $(G_n(a,b;r,s))_{n\geqslant 0}$ where $a,b,r,s \in \mathbb{Z}$, satisfying the two following conditions:

(1) $G_0(a,b;r,s) = a, \quad G_1(a,b;r,s) = b,$

(2) $G_n(a,b;r,s) = rG_{n-1}(a,b;r,s) + sG_{n-2}(a,b;r,s), \quad n \geqslant 2.$

Here note that $G_n(0,1;1,1) = F_n$ are the Fibonacci numbers, $G_n(2,1;1,1) = L_n$ are the Lucas numbers, $G_n(0,1;2,1) = P_n$ are the Pell numbers and $G_n(0,1;1,2) = J_n$ are the Jacobsthal numbers. For convenience, some of values of F_n, L_n, P_n and J_n for $0 \leqslant n \leqslant 10$ are determined in the following:

n	0	1	2	3	4	5	6	7	8	9	10	…
F_n	0	1	1	2	3	5	8	13	21	34	55	…
P_n	0	1	2	5	12	29	70	169	408	985	2378	…
J_n	0	1	1	3	5	11	21	43	85	171	341	…
L_n	2	1	3	4	7	11	18	29	47	76	123	…

We remind the reader of the famous Binet formula that can be used to calculate the Fibonacci and Lucas numbers (see [19]):

$$\sqrt{5}F_n = \phi^n - \Phi^n \quad \text{and} \quad L_n = \phi^n + \Phi^n \quad (n = 0,1,2,\ldots),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\Phi = \frac{1-\sqrt{5}}{2}$ is the golden ratio conjugate.

In the sequel, we introduce transformations that operate on integer sequences. An example of such a transformation that is widely used in the study of integer sequences is the so-called *binomial transform*, which associates to the sequence α with general term α_i the sequence $\hat{\alpha}$ with general term $\hat{\alpha}_i$, where

$$\hat{\alpha}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \alpha_k.$$

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This transformation is invertible. The *inverse binomial transform* $\check{\alpha} = (\check{\alpha}_i)_{i\geqslant 0}$ is given by

$$\check{\alpha}_i = \sum_{k=0}^{i} \binom{i}{k} \alpha_k.$$

Therefore, we have $\hat{\check{\alpha}} = \check{\hat{\alpha}} = \alpha$ [10, Lemma 2.1].

Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two sequences starting with $\alpha_0 = \beta_0$. We define an infinite dimensional matrix $P^{[a,b,c]}_{\alpha,\beta}(\infty) = [P_{i,j}]_{i,j\geqslant 0}$ associated with parameters a,b,c and sequences α and β, by setting:

$$P_{i,0} = \alpha_i \quad \text{and} \quad P_{0,i} = \beta_i \quad \text{for } i \geqslant 0,$$

and

$$P_{i,j} = aP_{i,j-1} + bP_{i-1,j-1} + cP_{i-1,j} \quad \text{for } i,j \geqslant 1.$$

We call the infinite dimensional matrix $P^{[a,b,c]}_{\alpha,\beta}(\infty)$ the *Pascal-like matrix* associated with parameters a,b,c and sequences α and β. Two special cases arise frequently in practice:

- $[a,b,c] = [1,0,1]$. In this case, we call the infinite dimensional matrix $P_{\alpha,\beta}(\infty) := P^{[1,0,1]}_{\alpha,\beta}(\infty)$ the *generalized Pascal triangle* associated with α and β (see [1]).
- $[a,b,c] = [0,1,1]$. In this case, we call the infinite dimensional matrix $S_{\alpha,\beta}(\infty) := P^{[0,1,1]}_{\alpha,\beta}(\infty)$ the 7-*matrix* associated with α and β (see [6]). Especially, we put

  $$L(\infty) := S_{(1,1,1,\ldots),(1,0,0,\ldots)}(\infty),$$

  which is called the *Pascal lower triangular*, and $U(\infty) := L(\infty)^T$, where A^T signifies the transpose of a matrix A.

We recall that an infinite dimensional matrix $T(\infty) = [T_{i,j}]_{i,j\geqslant 0}$ is said to be *Toeplitz* if

$$T_{i,j} = T_{k,l}, \quad \text{whenever } i-j = k-l.$$

When $\alpha = (T_{i,0})_{i\geqslant 0}$ and $\beta = (T_{0,j})_{j\geqslant 0}$, we will write $T_{\alpha,\beta}(\infty)$ for the Toeplitz matrix $T(\infty)$. We also denote by $T_{\alpha,\beta}(n)$, the submatrix of $T_{\alpha,\beta}(\infty)$ consisting of the entries in its first $n+1$ rows and columns. Such matrices arise in many applications (for example, see [8]).

An $(n+1)\times(n+1)$ matrix of the following form:

$$\left[\begin{array}{c|c} A & B \\ \hline C & P^{[a,b,c]}_{\alpha,\beta}(n-r) \end{array}\right] \qquad \left(\text{resp. } \left[\begin{array}{c|c} A & B \\ \hline C & T_{\alpha,\beta}(n-r) \end{array}\right]\right),$$

where A is an $r\times r$ matrix, B is an $r\times(n-r+1)$ matrix, and C is an $(n-r+1)\times r$ matrix, is called a *quasi Pascal-like* (resp. *quasi Toeplitz*) matrix.

We introduce some more notation. We will denote by $\lceil x \rceil$ the smallest integer greater than or equal to x. Also, the transpose of A will be denoted by A^T. Given an arbitrary sequence $\alpha = (\alpha_i)_{i\geqslant 0}$, we put $\tilde{\alpha} = (\tilde{\alpha}_i)_{i\geqslant 0}$ where $\tilde{\alpha}_i = (-1)^i \alpha_i$ for all $i \geqslant 0$. Note that we index matrices starting at $(0,0)$. Sometimes, we will study together the Fibonacci and Lucas sequences, and, in order to unify our treatment, we introduce the following useful notations. For $\varepsilon \in \{+,-\}$, we let $F^{\varepsilon}_n = F_n$ if $\varepsilon = +$, and $F^{\varepsilon}_n = L_n$ if $\varepsilon = -$.

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3 Tridiagonal Matrices

We recall that a matrix is called *tridiagonal* if all of its non-zero entries appear only on the main diagonal, superdiagonal or subdiagonal. There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For instance, consider the families of infinite dimensional tridiagonal matrices of the following form:

$$T_{\omega}^{\varepsilon}(\infty)=\begin{pmatrix} F_2^{\varepsilon} & \omega_0 & 0 & 0 & 0 & \cdots \\ -\omega_0^{-1} & 1 & \omega_1 & 0 & 0 & \cdots \\ 0 & -\omega_1^{-1} & 1 & \omega_2 & 0 & \cdots \\ 0 & 0 & -\omega_2^{-1} & 1 & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{pmatrix}$$

where $\omega=(\omega_i)_{i\geqslant 0}$ with $\omega_i\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. These matrices are described in [11] (also in [3, 4], when $\varepsilon=+$, $\omega_i=\sqrt{-1}$, in [16] when $\varepsilon=+$, $\omega_i=1$, in [2] when $\varepsilon=-$, $\omega_i=\sqrt{-1}$, and in [20] when $\varepsilon=+$, $\omega_i=\sqrt{-1}$). Moreover, one can easily check that (for example by induction) the leading principal minors of $T_{\omega}^{\varepsilon}(n)$, the $(n+1)\times(n+1)$ upper left corner matrix of $T_{\omega}^{\varepsilon}(\infty)$, form the subsequence $(F_{n+2}^{\varepsilon})_{n\geqslant 0}$ of the Fibonacci or Lucas sequence.

Also, the sequence of leading principal minors of the infinite dimensional Toeplitz matrices:

$$T_{(3,t,0,\ldots),(3,t,0,\ldots)}(\infty)=\begin{pmatrix} 3 & t & 0 & 0 & 0 & \cdots \\ t & 3 & t & 0 & 0 & \cdots \\ 0 & t & 3 & t & 0 & \cdots \\ 0 & 0 & t & 3 & t & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $t=\pm 1$, is the subsequence $(F_{2n+4})_{n\geqslant 0}$ of Fibonacci sequence (see [16, 17]).

In [5], Cahill and Narayan constructed other families of tridiagonal matrices whose leading principal minors generate any linear subsequence $(F_{a(n+1)+b}^{\varepsilon})_{n\geqslant 0}$ $(a\in\mathbb{Z}^+, b\in\mathbb{N})$ of the Fibonacci or Lucas sequence. Indeed, they proved the following theorem.

Theorem 3.1 (Cahill-Narayan [4]). *Let a be a nonnegative integer and b a natural number. Suppose that*

$$\varphi_{a,b}=\left\lceil\frac{F_{2a+b}^{\varepsilon}}{F_{a+b}^{\varepsilon}}\right\rceil,\quad \psi_{a,b}=\sqrt{\varphi_{a,b}F_{a+b}^{\varepsilon}-F_{2a+b}^{\varepsilon}}\quad \text{and}\quad \alpha=(L_a,\sqrt{(-1)^a},\sqrt{(-1)^a},\ldots),$$

and put

$$T_{a,b}^{\varepsilon}(\infty)=\left[\begin{array}{cc|ccc} F_{a+b}^{\varepsilon} & \psi_{a,b} & 0 & 0 & \ldots \\ \psi_{a,b} & \varphi_{a,b} & \sqrt{(-1)^a} & 0 & \ldots \\ \hline 0 & \sqrt{(-1)^a} & & & \\ 0 & 0 & & T_{\alpha,\alpha}(\infty) & \\ \vdots & \vdots & & & \end{array}\right].$$

Then, we have

$$\det T_{a,b}^{\varepsilon}(n)=F_{a(n+1)+b}^{\varepsilon},\quad n=0,1,2,\ldots,$$

where $T_{a,b}^{\varepsilon}(n)$ is the $(n+1)\times(n+1)$ upper left corner matrix of $T_{a,b}^{\varepsilon}(\infty)$.

Another family of matrices that satisfies Theorem 3.1 can be found by choosing the negative root for all of the super-diagonal and sub-diagonal entries.

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4 Matrices With Recursive Entries

In this section, we first present a *factorization* of the generalized Pascal triangle $P_{\alpha,\beta}(n)$ (resp. the 7-matrix $S_{\alpha,\beta}(n)$) associated with the arbitrary sequences α and β (with common first term), as a product of a unipotent lower triangular matrix, a Toeplitz matrix and a unipotent upper triangular matrix (resp. as a product of a unipotent lower triangular matrix and a Toeplitz matrix). In fact, we obtain a *connection* between generalized Pascal triangles (resp. 7-matrices) associated with α and β, and Toeplitz matrices. More precisely, we have proved:

Theorem 4.1. (see [9, 10, 18]) *Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two sequences with $\alpha_0 = \beta_0$. Then we have the following factorizations:*

$$P_{\alpha,\beta}(n) = L(n)\cdot T_{\hat{\alpha},\hat{\beta}}(n)\cdot U(n) \quad \text{and} \quad S_{\alpha,\beta}(n) = L(n)\cdot T_{\hat{\alpha},\beta}(n),$$

or equivalently

$$T_{\alpha,\beta}(n) = L(n)^{-1}\cdot P_{\check{\alpha},\check{\beta}}(n)\cdot U(n)^{-1} \quad \text{and} \quad T_{\alpha,\beta}(n) = L(n)^{-1}\cdot S_{\check{\alpha},\beta}(n).$$

In particular, the matrices $T_{\alpha,\beta}(\infty)$, $P_{\check{\alpha},\check{\beta}}(\infty)$ and $S_{\check{\alpha},\beta}(\infty)$ are equimodular.

In [11], we proved the following two theorems.

Theorem 4.2. *Let $\alpha^{\varepsilon} = (\alpha_i)_{i\geqslant 0}$ be the arithmetic sequence with $\alpha_0 = 1$ and common difference ± 1 if $\varepsilon = +$, and $\pm\sqrt{-1}$ if $\varepsilon = -$, and let $\beta^{\varepsilon} = (\beta_i)_{i\geqslant 0}$ be a 2-periodic sequence given by*

$$\beta^{\varepsilon} = \begin{cases} (1,0,1,0,\ldots) & \text{if} \quad \varepsilon = +, \\ (1,\ \omega 2\sqrt{-1},\ 1,\ \omega 2\sqrt{-1},\ldots) & \text{if} \quad \varepsilon = -; \end{cases}$$

where $\omega \in \{+,-\}$. Then there holds

$$\det S_{\alpha^{\varepsilon},\beta^{\varepsilon}}(n) = F^{\varepsilon}_{n+1}.$$

Theorem 4.3. *Suppose $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ are two arithmetic sequences with $\alpha_0 = \beta_0 = 1$ and common differences $-c^{-1}$ and $c \neq 0$, respectively. Then we have $\det P_{\alpha,\beta}(n) = F_{n+2}$.*

We also proved the following result in [10].

Theorem 4.4. *Let a be a nonnegative integer and b a natural number. Suppose that*

$$\varphi_{a,b} = \left\lceil \frac{F^{\varepsilon}_{2a+b}}{F^{\varepsilon}_{a+b}} \right\rceil \quad \text{and} \quad \psi_{a,b} = \sqrt{\varphi_{a,b}F^{\varepsilon}_{a+b} - F^{\varepsilon}_{2a+b}},$$

and $\alpha = (\alpha_i)_{i\geqslant 0}$ is an arithmetic sequence with $\alpha_i = L_a + i\sqrt{(-1)^a}$. Then, the leading principal minors of the following infinite dimensional quasi-Pascal matrix:

$$\left[\begin{array}{cc|ccc} F^{\varepsilon}_{a+b} & \psi_{a,b} & 0 & 0 & \cdots \\ \psi_{a,b} & \phi_{a,b} & \sqrt{(-1)^a} & \sqrt{(-1)^a} & \cdots \\ \hline 0 & \sqrt{(-1)^a} & & & \\ 0 & \sqrt{(-1)^a} & & P_{\alpha,\alpha}(\infty) & \\ \vdots & \vdots & & & \end{array}\right]$$

form the subsequence $(F^{\varepsilon}_{(n+1)a+b})_{n\geqslant 0}$ of the Fibonacci or Lucas sequence.

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In the following theorem (see [9]) we show that there exist some infinite families of new Toeplitz matrices $T_{\alpha,\beta}(\infty)$ and Pascal-like matrices $S_{\alpha,\beta}(\infty)$ and $P_{\alpha,\beta}(\infty)$ whose leading principal minors form the Fibonacci subsequence $(F_{n+1})_{n\geqslant 0}$.

Theorem 4.5 ([9])**.** *Let A_1, A_2, B_1 and B_2 be integers with $A_2, B_2 \neq 0$. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two integer sequences satisfying $\alpha_0 = \beta_0 = \gamma$ and linear recurrences*

$$\alpha_i = A_1\alpha_{i-1} + A_2\alpha_{i-2} \quad \textit{and} \quad \beta_i = B_1\beta_{i-1} + B_2\beta_{i-2} \quad \textit{for all} \;\; i \geqslant 2,$$

of order 2. Then, for any nonnegative integer n, we have

(a) $\det T_{\alpha,\beta}(n) = F_{n+1}$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the conditions* $(1)-(8)$.

(b) $\det S_{\alpha,\beta}(n) = F_{n+1}$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the conditions* $(1')-(8')$.

(c) $\det P_{\alpha,\beta}(n) = F_{n+1}$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the conditions* $(1'')-(8'')$.

Here, $C = 2 + c$ and $\bar{C} = 2 - c$ for a constant c.

	γ	α_1	β_1	A_1	A_2	B_1	B_2		γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	1	0	$c+1$	$-c$	1	c	c	(5)	1	$c+1$	0	c	c	$-c$	1
(2)	1	0	$c-1$	c	-1	$-\bar{C}$	c	(6)	1	$c-1$	0	$-\bar{C}$	c	c	-1
(3)	1	0	$c-1$	$-c$	1	c	$-c$	(7)	1	$c-1$	0	c	$-c$	$-c$	1
(4)	1	0	$1-c$	$-c$	-1	$\bar{C}$	c	(8)	1	$1-c$	0	$\bar{C}$	c	$-c$	-1
$(1')$	1	1	$c+1$	$\bar{C}$	c	c	c	$(5')$	1	C	0	C	-1	$-c$	1
$(2')$	1	1	$c-1$	C	$-C$	$-\bar{C}$	c	$(6')$	1	c	0	c	1	c	-1
$(3')$	1	1	$c-1$	$\bar{C}$	c	c	$-c$	$(7')$	1	c	0	C	$-(2c+1)$	$-c$	1
$(4')$	1	1	$1-c$	$\bar{C}$	$-\bar{C}$	$\bar{C}$	c	$(8')$	1	$\bar{C}$	0	$4-c$	$2c-3$	$-c$	-1
$(1'')$	1	1	C	$\bar{C}$	c	C	-1	$(5'')$	1	C	1	C	-1	$\bar{C}$	c
$(2'')$	1	1	c	C	$-C$	c	1	$(6'')$	1	c	1	c	1	C	$-C$
$(3'')$	1	1	c	$\bar{C}$	c	C	$-(2c+1)$	$(7'')$	1	c	1	C	$-(2c+1)$	$\bar{C}$	c
$(4'')$	1	1	$\bar{C}$	$\bar{C}$	$-\bar{C}$	$4-c$	$2c-3$	$(8'')$	1	$\bar{C}$	1	$4-c$	$2c-3$	$\bar{C}$	$-\bar{C}$

Although in Theorem 4.5, we have restricted ourselves to the integer values, but it is worth pointing out that the result still holds for *all* values of c (not necessarily an integer) in above conditions. Moreover, note that the matrices constructed by the conditions in the left column are transpose of the matrices constructed by the corresponding conditions in the right column.

An Exotic Example. As a special case of Theorem 4.5(b-$1'$), we see that the Fibonacci numbers provide an exotic example. More precisely, if we take $c = 1$, then we obtain

$$\det S_{(F_1,F_2,F_3,\ldots),\,(F_2,F_3,F_4,\ldots)}(n) = \begin{pmatrix} F_1 = F_2 & F_3 & F_4 & F_5 & \cdots & F_{n+2} \\ F_2 & F_4 & F_5 & F_6 & \cdots & F_{n+3} \\ F_3 & * & F_6 & F_7 & \cdots & F_{n+4} \\ F_4 & * & * & F_8 & \cdots & F_{n+5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{n+1} & * & * & * & \cdots & F_{2n+2} \end{pmatrix} = F_{n+1}.$$

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In the similar manner, one can get some infinite families of Toeplitz matrices, 7-matrices or generalized Pascal triangles whose leading principal minors form the Lucas sequence $(L_n)_{n\geqslant 0}$ (see Corollary 3.4 in [9]).

Theorem 4.6 ([9])**.** *Let A_1, A_2, B_1 and B_2 be integers. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two integer sequences satisfy $\alpha_0 = \beta_0 = \gamma$ and linear recurrences*

$$\alpha_i = A_1\alpha_{i-1} + A_2\alpha_{i-2} \quad \textit{and} \quad \beta_i = B_1\beta_{i-1} + B_2\beta_{i-2} \quad \textit{for all} \;\; i \geqslant 2.$$

Then, for any nonnegative integer n, we have:

(a) $\det T_{\alpha,\beta}(n) = L_n$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the conditions* $(1) - (32)$.

(b) $\det S_{\alpha,\beta}(n) = L_n$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the conditions* $(1') - (32')$.

(c) $\det P_{\alpha,\beta}(n) = L_n$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the conditions* $(1'') - (32'')$.

	γ	α_1	β_1	A_1	A_2	B_1	B_2		γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	2	3	1	0	2	−5	4	(17)	2	1	3	−5	4	0	2
(2)	2	3	1	4	−4	−9	6	(18)	2	1	3	−9	6	4	−4
(3)	2	3	1	−1	4	10	−6	(19)	2	1	3	10	−6	−1	4
(4)	2	3	1	3	−2	6	−4	(20)	2	1	3	6	−4	3	−2
(5)	2	3	1	6	−8	−3	2	(21)	2	1	3	−3	2	6	−8
(6)	2	3	1	2	−2	1	0	(22)	2	1	3	1	0	2	−2
(7)	2	3	1	−3	8	4	−2	(23)	2	1	3	4	−2	−3	8
(8)	2	3	1	1	2	0	0	(24)	2	1	3	0	0	1	2
(9)	2	−3	−1	−2	−2	−1	0	(25)	2	−1	−3	−1	0	−2	−2
(10)	2	−3	−1	−6	−8	3	2	(26)	2	−1	−3	3	2	−6	−8
(11)	2	−3	−1	0	2	5	4	(27)	2	−1	−3	5	4	0	2
(12)	2	−3	−1	−4	−4	9	6	(28)	2	−1	−3	9	6	−4	−4
(13)	2	−3	−1	1	4	−10	−6	(29)	2	−1	−3	−10	−6	1	4
(14)	2	−3	−1	−3	−2	−6	−4	(30)	2	−1	−3	−6	−4	−3	−2
(15)	2	−3	−1	3	8	−4	−2	(31)	2	−1	−3	−4	−2	3	8
(16)	2	−3	−1	−1	2	0	0	(32)	2	−1	−3	0	0	−1	2

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	γ	α_1	β_1	A_1	A_2	B_1	B_2		γ	α_1	β_1	A_1	A_2	B_1	B_2
$(1')$	2	5	1	2	1	-5	4	$(17')$	2	3	3	-3	8	0	2
$(2')$	2	5	1	6	-9	-9	6	$(18')$	2	3	3	-7	14	4	-4
$(3')$	2	5	1	1	4	10	-6	$(19')$	2	3	3	12	-17	-1	4
$(4')$	2	5	1	5	-6	6	-4	$(20')$	2	3	3	8	-11	3	-2
$(5')$	2	5	1	8	-15	-3	2	$(21')$	2	3	3	-1	4	6	-8
$(6')$	2	5	1	4	-5	1	0	$(22')$	2	3	3	3	-2	2	-2
$(7')$	2	5	1	-1	10	4	-2	$(23')$	2	3	3	6	-7	-3	8
$(8')$	2	5	1	3	0	0	0	$(24')$	2	3	3	2	-1	1	2
$(9')$	2	-1	-1	0	-1	-1	0	$(25')$	2	1	-3	1	0	-2	-2
$(10')$	2	-1	-1	-4	-3	3	2	$(26')$	2	1	-3	5	-2	-6	-8
$(11')$	2	-1	-1	2	1	5	4	$(27')$	2	1	-3	7	-2	0	2
$(12')$	2	-1	-1	-2	-1	9	6	$(28')$	2	1	-3	11	-4	-4	-4
$(13')$	2	-1	-1	3	2	-10	-6	$(29')$	2	1	-3	-8	3	1	4
$(14')$	2	-1	-1	-1	0	-6	-4	$(30')$	2	1	-3	-4	1	-3	-2
$(15')$	2	-1	-1	5	4	-4	-2	$(31')$	2	1	-3	-2	1	3	8
$(16')$	2	-1	-1	1	2	0	0	$(32')$	2	1	-3	2	-1	-1	2
$(1'')$	2	5	3	2	1	-3	8	$(17'')$	2	3	5	-3	8	2	1
$(2'')$	2	5	3	6	-9	-7	14	$(18'')$	2	3	5	-7	14	6	-9
$(3'')$	2	5	3	1	4	12	-17	$(19'')$	2	3	5	12	-17	1	4
$(4'')$	2	5	3	5	-6	8	-11	$(20'')$	2	3	5	8	-11	5	-6
$(5'')$	2	5	3	8	-15	-1	4	$(21'')$	2	3	5	-1	4	8	-15
$(6'')$	2	5	3	4	-5	3	-2	$(22'')$	2	3	5	3	-2	4	-5
$(7'')$	2	5	3	-1	10	6	-7	$(23'')$	2	3	5	6	-7	-1	10
$(8'')$	2	5	3	3	0	2	-1	$(24'')$	2	3	5	2	-1	3	0
$(9'')$	2	-1	1	0	-1	1	0	$(25'')$	2	1	-1	1	0	0	-1
$(10'')$	2	-1	1	-4	-3	5	-2	$(26'')$	2	1	-1	5	-2	-4	-3
$(11'')$	2	-1	1	2	1	7	-2	$(27'')$	2	1	-1	7	-2	2	1
$(12'')$	2	-1	1	-2	-1	11	-4	$(28'')$	2	1	-1	11	-4	-2	-1
$(13'')$	2	-1	1	3	2	-8	3	$(29'')$	2	1	-1	-8	3	3	2
$(14'')$	2	-1	1	-1	0	-4	1	$(30'')$	2	1	-1	-4	1	-1	0
$(15'')$	2	-1	1	5	4	-2	1	$(31'')$	2	1	-1	-2	1	5	4
$(16'')$	2	-1	1	1	2	2	-1	$(32'')$	2	1	-1	2	-1	1	2

Again, it is worth mentioning here that the matrices constructed by conditions in the left column are the transpose of the matrices constructed by corresponding conditions in the right column.

In what follows, we will focus our attention on some determinant representations of sequences $(G_n(0,1;r,s))_{n\geqslant 0}$ where $r=1$ or $s=1$. Especially, we will obtain some infinite families of integer matrices whose leading principal minors form the Fibonacci, Pell and Jacobsthal sequence (taking $r=1,2$ in Theorem 4.7 and $s=2$ in Theorem 4.8, respectively).

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Theorem 4.7 ([9]). *Let A_1, A_2, B_1, B_2 and r be integers with $r \geqslant 1$. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two integer sequences satisfy $\alpha_0 = \beta_0 = \gamma$ and linear recurrences*

$$\alpha_i = A_1\alpha_{i-1} + A_2\alpha_{i-2} \quad \text{and} \quad \beta_i = B_1\beta_{i-1} + B_2\beta_{i-2} \quad \text{for all } i \geqslant 2.$$

Then, for any nonnegative integer n, there hold:

(a) $\det T_{\alpha,\beta}(n) = G_n(0,1;r,1)$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+r$	0	c	0
(2)	0	1	-1	$c+r$	-2	c	-2
(3)	0	-1	1	c	0	$c+r$	0
(4)	0	-1	1	c	-2	$c+r$	-2

(b) $\det S_{\alpha,\beta}(n) = G_n(0,1;r,1)$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+r+2$	$-c-r-1$	c	0
(2)	0	1	-1	$c+r+2$	$-c-r-3$	c	-2
(3)	0	-1	1	$c+2$	$-c-1$	$c+r$	0
(4)	0	-1	1	$c+2$	$-c-3$	$c+r$	-2

(c) $\det P_{\alpha,\beta}(n) = G_n(0,1;r,1)$ *iff γ, α_1, β_1, A_1, A_2, B_1 and B_2 satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+r+2$	$-c-r-1$	$c+2$	$-c-1$
(2)	0	1	-1	$c+r+2$	$-c-r-3$	$c+2$	$-c-3$
(3)	0	-1	1	$c+2$	$-c-1$	$c+r+2$	$-c-r-1$
(4)	0	-1	1	$c+2$	$-c-3$	$c+r+2$	$-c-r-3$

Theorem 4.8 ([9]). *Let A_1, A_2, B_1 and B_2 be integers and let s be a prime number. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two integer sequences satisfy $\alpha_0 = \beta_0 = \gamma$ and linear recurrences*

$$\alpha_i = A_1\alpha_{i-1} + A_2\alpha_{i-2} \quad \text{and} \quad \beta_i = B_1\beta_{i-1} + B_2\beta_{i-2} \quad \text{for all } i \geqslant 2.$$

Then, for any nonnegative integer n, we have

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(a) $\det T_{\alpha,\beta}(n) = G_n(0,1;1,s)$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+1$	0	c	$s-1$
(2)	0	1	-1	$c+1$	-2	c	$-s-1$
(3)	0	1	-1	$c+1$	$s-1$	c	0
(4)	0	1	-1	$c+1$	$-s-1$	c	-2
(5)	0	-1	1	c	$s-1$	$c+1$	0
(6)	0	-1	1	c	$-s-1$	$c+1$	-2
(7)	0	-1	1	c	0	$c+1$	$s-1$
(8)	0	-1	1	c	-2	$c+1$	$-s-1$

(b) $\det S_{\alpha,\beta}(n) = G_n(0,1;1,s)$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+3$	$-c-2$	c	$s-1$
(2)	0	1	-1	$c+3$	$-c-4$	c	$-s-1$
(3)	0	1	-1	$c+3$	$s-c-3$	c	0
(4)	0	1	-1	$c+3$	$-s-c-3$	c	-2
(5)	0	-1	1	$c+2$	$s-c-2$	$c+1$	0
(6)	0	-1	1	$c+2$	$-s-c-2$	$c+1$	-2
(7)	0	-1	1	$c+2$	$-c-1$	$c+1$	$s-1$
(8)	0	-1	1	$c+2$	$-c-3$	$c+1$	$-s-1$

(c) $\det P_{\alpha,\beta}(n) = G_n(0,1;1,s)$ *iff* γ, α_1, β_1, A_1, A_2, B_1 *and* B_2 *satisfy one of the following conditions, where c is a constant.*

	γ	α_1	β_1	A_1	A_2	B_1	B_2
(1)	0	1	-1	$c+3$	$-c-2$	$c+2$	$s-c-2$
(2)	0	1	-1	$c+3$	$-c-4$	$c+2$	$-s-c-2$
(3)	0	1	-1	$c+3$	$s-c-3$	$c+2$	$-c-1$
(4)	0	1	-1	$c+3$	$-s-c-3$	$c+2$	$-c-3$
(5)	0	-1	1	$c+2$	$s-c-2$	$c+3$	$-c-2$
(6)	0	-1	1	$c+2$	$-s-c-2$	$c+3$	$-c-4$
(7)	0	-1	1	$c+2$	$-c-1$	$c+3$	$s-c-3$
(8)	0	-1	1	$c+2$	$-c-3$	$c+3$	$-s-c-3$

Although in Theorems 4.7 and 4.8, we have restricted ourselves to the integer values, but we should mention that the necessary part still holds for *all* values of c (not necessarily an integer) in the above conditions. Also, as before some of the matrices constructed by the above conditions are the transpose of each other.

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There are also scattered results in the literature showing that certain (quasi-)Toeplitz matrices are determinant representations of some subsequences of Fibonacci sequence. For instance, consider the quasi Toeplitz matrix

$$T(n) = \left[\begin{array}{c|c} 1 & B \\ \hline C & T_{(2,1,1,1,\ldots),(2,-1,0,0,\ldots)}(n-1) \end{array}\right],$$

where $C = (1,1,\ldots,1)^T$ and $B = (-1,0,0,\ldots,0)$. Then, we have $\det T(n) = F_{2n+2}$ (see [3]). Also, the leading principal minors of Toeplitz matrices:

$$Q = T_{(2,1,1,1,\ldots),(2,t,0,0,\ldots)}(\infty),$$

where $t = \pm 1$, form the sequence $(F_{n+3})_{n\geqslant 0}$ for $t = 1$ and the sequence $(F_{2n+3})_{n\geqslant 0}$ for $t = -1$ ([3, Examples 1, 2]).

Two other interesting examples are due to Griffin, Stuart and Tsatsomeros in [7]. In fact, the golden ratio and its conjugate are used in the structures of these examples.

Theorem 4.9 ([7])**.** *For each nonnegative integer n, let*

$$P_n = T_{(1,\Phi,\Phi,\ldots),(1,\phi,\phi,\ldots)}(n), \ \ \textit{and} \ \ Q_n = T_{(0,-\Phi,-\Phi,\ldots),(0,-\phi,-\phi,\ldots)}(n).$$

Then we have

$$\det P_n = F_{n+2}, \ \ \textit{and} \ \ \det Q_n = F_n.$$

In the sequel we give two other results. Our first result is the following corollary which can be obtained using Theorem 4.1, Theorem 4.9, the results collected in Section 2 and what observed above.

Corollary 4.1. *Let n be a nonnegative integer, $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two sequences, and let d_n be the nth leading principal minor of $P_{\alpha,\beta}(\infty)$. Then, the following hold:*

(1) *If $\alpha_i = \beta_i = 1 + i\sqrt{-1}$, then $d_n = F_{n+2}$.*

(2) *If $\alpha_i = \beta_i = 3 - i$, then $d_n = F_{2n+4}$.*

(3) *If $\alpha_i = \beta_i = 3 + i$, then $d_n = F_{2n+4}$.*

(4) *If $\alpha_i = 1 + i$ and $\beta_i = 1 - i$, then $d_n = F_{n+2}$.*

(5) *If $\alpha_i = 2 - i$ and $\beta_i = 2^i + 1$, then $d_n = F_{2n+3}$.*

(6) *If $\alpha_i = 2 + i$ and $\beta_i = 2^i + 1$, then $d_n = F_{n+3}$.*

(7) *If $\alpha_i = (2^i - 1)\Phi + 1$ and $\beta_i = (2^i - 1)\phi + 1$, then $d_n = F_{n+2}$.*

(8) *If $\alpha_i = (1 - 2^i)\Phi$ and $\beta_i = (1 - 2^i)\phi$, then $d_n = F_n$.*

Another result is the following theorem, which is taken from [12].

Theorem 4.10. *Let $\alpha = (F_{i+1})_{i\geqslant 0}$. Then, for each nonnegative integer n, $\det T_{\tilde{\alpha},\alpha}(n) = F_{n+2}$.*

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Notice that, using the definition, if $\alpha = (F_{i+1})_{i\geqslant 0} = (1,1,2,3,5,8,\ldots)$, then

$$\begin{cases} \breve{\alpha} = (F_{2i+1})_{i\geqslant 0} = (1,2,5,13,34,\ldots) & \text{(A001519 in [15])} \\ \check{\breve{\alpha}} = (F_{i-1})_{i\geqslant 0} = (1,0,1,1,2,3,5,\ldots) & \text{(A212804 in [15])} \end{cases}$$

and by Proposition 4.10 and Theorem 4.1, the following infinite dimensional integer matrices:

$$T_{\breve{\alpha},\alpha}(\infty) = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdot \\ -1 & 1 & 1 & 2 & \cdot \\ 2 & -1 & 1 & 1 & \cdot \\ -3 & 2 & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad P_{\check{\breve{\alpha}},\breve{\alpha}}(\infty) = \begin{pmatrix} 1 & 2 & 5 & 13 & \cdot \\ 0 & 2 & 7 & 20 & \cdot \\ 1 & 3 & 10 & 30 & \cdot \\ 1 & 4 & 14 & 44 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

and

$$S_{\check{\breve{\alpha}},\alpha}(\infty) = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdot \\ 0 & 2 & 3 & 5 & \cdot \\ 1 & 2 & 5 & 8 & \cdot \\ 1 & 3 & 7 & 13 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

are $(F_{n+2})_{n\geqslant 0}$-equimodular.

In what follows, we will consider another class of matrices in which rows and columns (except for the first row and column) are obtained from a non-homogeneous recurrence relation. In [10], we obtained two matrices of this kind whose leading principal minors form the Fibonacci subsequence $(F_{n+1})_{n\geqslant 0}$. The first example was the following:

$$A(\infty) = [A_{i,j}]_{i,j\geqslant 0} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & 1 & \cdots \\ 1 & 4 & 6 & 6 & \cdots \\ 1 & 7 & 14 & 20 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Indeed, the leading principal minors of $A(\infty)$ form the Fibonacci subsequence $(F_{n+1})_{n\geqslant 0}$ (see [10]). This matrix is constructed as follows. The first row and column are the constant sequence $(1,1,1,\ldots)$. The remaining entries $A_{i,j}$ are obtained from the following non-homogeneous recurrence relation:

$$A_{i,j} = A_{i,j-1} + A_{i-1,j} + i - j, \qquad i,j \geqslant 1.$$

Theorem 4.11 ([10]). *For all $a,b \in \mathbb{C}$, let $A(\infty) = [A_{i,j}]_{i,j\geqslant 0}$ be an infinite dimensional matrix whose entries satisfy the following non-homogeneous recurrence relation*

$$A_{i,j} = A_{i,j-1} + A_{i-1,j} + ai + bj,$$

for $i,j \geqslant 1$, and the initial conditions $A_{i,0} = A_{0,j} = 1$, $i,j \geqslant 0$. If $d_n = \det A(n)$, $n \geqslant 0$, then $d_n = F_{n+1}$ iff $(a,b) \in \{(1,-1),(-1,1)\}$.

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Similarly, we will consider the following two infinite dimensional matrices

$$B(\infty)=[B_{i,j}]_{i,j\geqslant 0}=\begin{pmatrix} 2 & 3 & 4 & 5 & \cdots \\ -3 & -4 & -6 & -9 & \cdots \\ -27 & -37 & -51 & -70 & \cdots \\ -125 & -170 & -231 & -313 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$C(\infty)=[C_{i,j}]_{i,j\geqslant 0}=\begin{pmatrix} 2 & 1 & 3 & -1 & \cdots \\ 1 & 1 & 2 & 0 & \cdots \\ -2 & -2 & -1 & -2 & \cdots \\ -1 & -10 & -9 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As a matter of fact, these matrices are constructed as follows:

- *The Matrix* $B(\infty)$. The first row and column is the sequence

  $$(B_{0,j})_{j\geqslant 0}=(2,3,4,5,,\ldots,B_{0,j}=j+2,\ldots),$$

  and

  $$(B_{i,0})_{i\geqslant 0}=(2,-3,-27,\ldots,B_{i,0}=4B_{i-1,0}+i^2-7i-5,\ldots),$$

  respectively. The remaining entries $B_{i,j}$ are obtained from the non-homogeneous recurrence relation:

  $$B_{i,j}=B_{i,j-1}+B_{i-1,j}-2(i+j), \qquad i,j\geqslant 1.$$

- *The Matrix* $C(\infty)$. The first row and column is the sequence

  $$(C_{0,j})_{j\geqslant 0}=(2,1,3,-1,7,-9,23,-41,\ldots,C_{0,j}=-C_{0,j-1}+2C_{0,j-2},\ldots),$$

  and

  $$(C_{i,0})_{i\geqslant 0}=(2,1,-2,-1,42,\ldots,C_{i,0}=3C_{i-1,0}+5(3^{i-1}-2i-1)/2,\ldots),$$

  respectively. The remaining entries $C_{i,j}$ are obtained from the non-homogeneous recurrence relation:

  $$C_{i,j}=C_{i-1,j-1}+C_{i-1,j}-2i, \qquad i,j\geqslant 1.$$

As before, we denote by $B(n)$ (resp. $C(n)$) the submatrix of $B(\infty)$ (resp. $C(\infty)$) consisting of the elements in its first $n+1$ rows and columns.

Theorem 4.12 ([13]). *With the notation defined above, we have* $\det B(n)=\det C(n)=L_n$.

In what follows, we introduce another family of (infinite dimensional) integer matrices $D(\infty)=[D_{i,j}]_{i,j\geqslant 0}$, whose entries obey an non-homogeneous recurrence relation. Actually, for two constants u,v and arbitrary sequences $\lambda=(\lambda_i)_{i\geqslant 0}$ and $\mu=(\mu_i)_{i\geqslant 0}$ with $\mu_0=0$, the first column and row of matrix $D(\infty)$ is the sequences

$$(D_{i,0})_{i\geqslant 0}=(\lambda_0,\lambda_1,\lambda_2,\ldots,D_{i,0}=\lambda_i,\ldots),$$

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and

$$(D_{0,j})_{j\geqslant 0} = (\lambda_0, \lambda_0+u, \lambda_0+2u, \ldots, D_{0,j} = \lambda_0 + ju, \ldots),$$

respectively, while the remaining entries $D_{i,j}$ $(i, j \geqslant 1)$ are obtained from the following non-homogeneous recurrence relation:

$$D_{i,j} = D_{i,j-1} + D_{i-1,j} - \lambda_{i-1} + \mu_i - \mu_{i-1} + (j-1)(v-u), \qquad i,j \geqslant 1.$$

We denote by $D(n)$ the submatrix of $D(\infty)$ consisting of the entries in its first $n+1$ rows and columns. The matrix $D(3)$ for example is then given by

$$D(3) = \begin{pmatrix} \lambda_0 & \lambda_0+u & \lambda_0+2u & \lambda_0+3u \\ \lambda_1 & \lambda_1+\mu_1+u & \lambda_1+2\mu_1+2u+v & \lambda_1+3\mu_1+3u+3v \\ \lambda_2 & \lambda_2+\mu_2+u & \lambda_2+2\mu_2+\mu_1+2u+2v & \lambda_2+3\mu_2+3\mu_1+3u+7v \\ \lambda_3 & \lambda_3+\mu_3+u & \lambda_3+2\mu_3+\mu_2+\mu_1+2u+3v & \lambda_3+3\mu_3+3\mu_2+4\mu_1+3u+12v \end{pmatrix}.$$

Theorem 4.13 ([14])**.** *Let $D(n)$ be defined as above and let c be a constant. In the case when $u = v = 1$ and $\lambda_i = (2^i-1)c+1$, we have the following statements:*

(1) *if* $\mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right)c - \frac{i(i-3)}{2}$, *then* $\det D(n) = F_{n+1}$.

(2) *if* $\mu_i = \left(\frac{5}{4}\cdot 3^i - 2^i - \frac{2i+1}{4}\right)c + \frac{5}{4}(3^i-1) + \frac{i}{2}$, *then* $\det D(n) = L_{n+1}$.

(3) *if* $\mu_i = i^2c - i^2 + 2i$, *then* $\det D(n) = J_{n+1}$.

(4) *if* $\mu_i = \left(2^{i+1} + \frac{(i+1)(i-4)}{2}\right)c + \frac{(5-i)i}{2}$, *then* $\det D(n) = P_{n+1}$.

5 Convolution Matrices

Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two arbitrary sequences. The *convolution* of sequences α and β, is the sequence $\gamma = (\gamma_i)_{i\geqslant 0}$, where $\gamma_i = \sum_{k=0}^{i} \alpha_k \beta_{i-k}$. The *convolution matrix* associated with sequences α and β is the infinite dimensional matrix $A = [A_{i,j}]_{i,j\geqslant 0}$ whose first column, i.e., $\mathrm{C}_0(A)$, is α and whose jth column $(j = 1,2,\ldots)$ is the convolution of two sequences $\mathrm{C}_{j-1}(A)$ (the $(j-1)$th column) and β. There are many well-known integer matrices which can be written as convolution matrices of some sequences. For instance, if $\alpha = (1,1,1,\ldots)$, $\beta = (1,0,0,\ldots)$ and $\gamma = (1,1,0,0,\ldots)$, then $P_{\alpha,\alpha}(\infty)$ is the convolution matrix of the sequences α and α, and the upper triangular matrix $U(\infty)$ is the convolution matrix of the sequences β and γ.

In [20], the authors showed that *any* sequence can be represented in terms of principal minors associated with a certain convolution matrix. In fact, they proved the following theorem:

Theorem 5.1 (see [20])**.** *Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be given sequences with $\beta_0 = 1$, and let $\xi = (\xi_i)_{i\geqslant 0} = (\xi_0, 1, 0, 0, \ldots)$. Let $A(\infty)$ be the following infinite dimensional matrix*

$$A(\infty) = \left(\begin{bmatrix} \text{Convolution sequence} \\ \text{of } \tilde{\alpha} \text{ and } \beta \end{bmatrix} \middle| \begin{bmatrix} \text{Convolution matrix} \\ \text{of } \beta \text{ and } \xi \end{bmatrix} \right).$$

Then $\det A(n) = \alpha_n$, for $n = 0,1,2,\ldots$, where $A(n)$ is the $(n+1)\times(n+1)$ upper left corner matrix of $A(\infty)$.

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One of the most interesting examples of such matrices is provided when we take $\alpha = (\alpha_i)_{i\geqslant 0}$, $\beta = (1,0,0,\dots)$ and $\gamma = (1,1,0,0,\dots)$. In this case, we obtain

$$A(\infty) = \left(\begin{array}{c|c} \tilde{\alpha}^T & U(\infty) \end{array} \right) = \begin{pmatrix} \alpha_0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -\alpha_1 & 0 & 1 & 2 & 3 & 4 & \cdots \\ \alpha_2 & 0 & 0 & 1 & 3 & 6 & \cdots \\ -\alpha_3 & 0 & 0 & 0 & 1 & 4 & \cdots \\ \alpha_4 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By expanding along the last row, we get $\det A(n) = \alpha_n$. In particular, if $\alpha = (F_i)_{i\geqslant 0}$ (resp. $(L_i)_{i\geqslant 0}$, $(P_i)_{i\geqslant 0}$ and $(J_i)_{i\geqslant 0}$), then $\det A(n) = F_n$ (resp. L_n, P_n and J_n). In a more general case, we can consider the infinite dimensional matrix:

$$B(\infty) = \left(\begin{array}{c|c} \tilde{\alpha}^T & V(\infty) \end{array} \right),$$

where $\alpha = (\alpha_i)_{i\geqslant 0}$ is a given sequence and $V(\infty)$ is an upper triangular matrix with 1s on the main diagonal. Again, by expanding the determinant of $B(n)$ with respect to the last row we get $\det B(n) = \alpha_n$, for $n = 0,1,2,\dots$, where $B(n)$ is the $(n+1)\times(n+1)$ upper left corner matrix of $B(\infty)$. Therefore, we obtain four infinite families of required matrices:

$$U_F(\infty) = \begin{pmatrix} F_0 & 1 & * & * & * & \cdots \\ -F_1 & 0 & 1 & * & * & \cdots \\ F_2 & 0 & 0 & 1 & * & \cdots \\ -F_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad U_L(\infty) = \begin{pmatrix} L_0 & 1 & * & * & * & \cdots \\ -L_1 & 0 & 1 & * & * & \cdots \\ L_2 & 0 & 0 & 1 & * & \cdots \\ -L_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$U_P(\infty) = \begin{pmatrix} P_0 & 1 & * & * & * & \cdots \\ -P_1 & 0 & 1 & * & * & \cdots \\ P_2 & 0 & 0 & 1 & * & \cdots \\ -P_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } U_J(\infty) = \begin{pmatrix} J_0 & 1 & * & * & * & \cdots \\ -J_1 & 0 & 1 & * & * & \cdots \\ J_2 & 0 & 0 & 1 & * & \cdots \\ -J_3 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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