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$*$-FUSION FRAMES IN HILBERT C^*-MODULES

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Abstract. Certain facts about frames are extended for the different kinds of frames in Hilbert C^*-modules. The paper presents the fusion frames with C^*-valued bounds in Hilbert C^*-modules where they are called $*$-fusion frames. Also, the relations between fusion frames and $*$-fusion frames in Hilbert C^*-modules are given such that $*$-fusion frames can be studied as fusion frames with different bounds.

1. Introduction

The theory of frames was rapidly generalized and various generalizations consisting of different vectors in Hilbert spaces were developed [3, 4]. It is well known that Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Let $\mathcal{A}$ be a C^*-algebra. An element a in $\mathcal{A}$ is called strictly nonzero if a is positive and invertible. A Hilbert C^*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module, is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ or $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, where $\mathcal{H}$ is a complete complex linear space, which is an algebraic (left) $\mathcal{A}$-module, and $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathcal{A}$, called an $\mathcal{A}$-inner product, possesses the following properties:

(1) $\langle f, f\rangle \geq 0$ for any $f \in \mathcal{H}$;
(2) $\langle f, f\rangle = 0$ if and only if $f = 0$;
(3) $\langle f, g\rangle^* = \langle g, f\rangle$ for any $f, g \in \mathcal{H}$;

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(4) $\langle \lambda f, h\rangle = \lambda\langle f, h\rangle$ whenever $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$;
(5) $\langle af+bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$ whenever $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

Throughout the paper, we fix the notations $\mathcal{A}$ and J for a given unital C^*-algebra and a finite or countably infinite index set, respectively. Also, the Hilbert $\mathcal{A}$-modules $\mathcal{H}$ is assumed to be finitely or countably generated Hilbert $\mathcal{A}$-module.

2. $*$-Fusion frames

In [3], the authors introduced the notion of a fusion frame for a given separable Hilbert space H as a family of ordered pairs $\{(F_j, \upsilon_j) : j \in J\}$ consisting of closed subspaces F_j of Hilbert space H and weights $\upsilon_j \in \mathbb{C}$, i.e., $\upsilon_j > 0$ for $j \in J$, satisfying

$$A||f||^2 \leq \sum_{j\in J} \upsilon_j^2 ||\pi_{F_j}(f)||^2 \leq B||f||^2,$$

for all $f \in H$ and some positive constants A and B where π_{F_j} is the orthogonal projection onto the subspace F_j, for all $j \in J$. And, the fusion frames for a finitely or countably generated Hilbert C^*-module have been defined [5]. In this extension, the fusion frame bounds are also read valued. Now, we introduce a new version of fusion frames with C^*-valued bounds in Hilbert C^*-modules that is called to be a $*$-fusion frame.

Definition 2.1. Let $\{F_j\}_{j\in J}$ be a family of orthogonally complemented submodules of Hilbert $\mathcal{A}$-module $\mathcal{H}$ and let $\{\upsilon_j\}_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each υ_j is a strictly nonzero element in the center of $\mathcal{A}$. The family of ordered pairs $\{(F_j, \upsilon_j)\}_{j\in J}$ is a $*$-fusion frame for $\mathcal{H}$ if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \sum_{j\in J} \upsilon_j^2 \langle \pi_{F_j}(f), \pi_{F_j}(f)\rangle \leq B\langle f, f\rangle B^*, \quad \forall f \in \mathcal{H}. \qquad (2.1)$$

The constants A and B are called lower and upper $*$-fusion frame bounds, respectively.

If $\lambda = A = B$, then the $*$-fusion frame $\{f_j\}_{j\in J}$ is said to be a λ-tight $*$-fusion frame. In the special case $A = B = 1_{\mathcal{A}}$, it is called Parseval $*$-fusion frame or a normalized $*$-fusion frame. In a Hilbert $\mathcal{A}$-module, the set of all normalized $*$-fusion frames and the set of all normalized fusion frames are precisely the same.

Remark 2.2. We mentioned that the set of all of fusion frames in Hilbert $\mathcal{A}$-modules can be considered as a subset of $*$-fusion frames. To illustrate this fact, let $\{(F_j, \upsilon_j)\}_{j\in J}$ be a fusion frame for Hilbert $\mathcal{A}$-module

$\mathcal{H}$ with real-valued fusion frame bounds A and B. Note that for $f \in \mathcal{H}$,

$$(\sqrt{A})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{A})1_{\mathcal{A}} \leq \sum_{j\in J} \upsilon_j^2 \langle \pi_{F_j}(f), \pi_{F_j}(f)\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every fusion frame for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ with real-valued bounds A and B is a $*$-fusion frame for $\mathcal{H}$ with $\mathcal{A}$-valued $*$-fusion frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$.

Example 2.3. Let $\{F_j\}_{j\in J}$ be a family of orthogonally complemented submodules of Hilbert $\mathcal{A}$-module $\mathcal{H}$ such that $\{(\pi_{F_j}, F_j)\}_{j\in J}$ is a $*$-g-frame for $\mathcal{H}$, i.e., a g-frame with C^*-valued bounds for $\mathcal{H}$. Then the sequence $\{(F_j, \upsilon_j)\}_{j\in J}$ is a $*$-fusion frame for $\mathcal{H}$, where $\upsilon_j = 1_{\mathcal{A}}$ for all $j \in J$.

Now, we consider the elementary definitions and properties of frames about new fusion frames. Similar to ordinary frames and [5], we can define the operators corresponding to a given $*$-fusion frame.

Definition 2.4. Let $\{(F_j, \upsilon_j)\}_{j\in J}$ be a $*$-fusion frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$. The pre-frame operator $\theta_{F,\upsilon}$ is

$$\theta_{F,\upsilon} : \mathcal{H} \longrightarrow l_2(\mathcal{H}), \qquad f \mapsto (\upsilon_j \pi_{F_j}(f))_{j\in J},$$

the adjoint operator of $\theta_{F,\upsilon}$ is

$$\theta^*_{F,\upsilon}(f_j)_{j\in J} = \sum_{j\in J} \upsilon_j \pi_{F_j}(f_j), \qquad \forall (f_j)_{j\in J} \in l_2(\mathcal{H}),$$

and the frame operator $S_{F,\upsilon}$ on $\mathcal{H}$ is defined by $S_{F,\upsilon}(f) = \sum_{j\in J} \upsilon_j^2 \pi_{F_j}(f)$.

The pre-frame operator corresponding to a given $*$-fusion frame has properties similar to fusion frame case. Let $\{(F_j, \upsilon_j)\}_{j\in J}$ be a $*$-fusion frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$. Then the pre-frame operator $\theta_{F,\upsilon}$ is an injective and adjointable map with closed rang. The adjoint operator $\theta^*_{F,\upsilon}$ is surjective. These properties are given by Theorem 2.9 [5] and the methods in the proof of Theorem 1.6 [2]. Moreover, Theorem 2.11 [5] and the methods in the proof of Theorem 1.8 [2] show that the frame operator $S_{F,\upsilon}$ associated to a given $*$-fusion frame $\{(F_j, \upsilon_j)\}_{j\in J}$ is positive, invertible, adjointable and $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$ when A and B are its $*$-fusion frame bounds.

Theorem 2.5. *Let $\{(F_j, \upsilon_j)\}_{j\in J}$ be a $*$-fusion frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$ with $*$-frame operator $S_{F,\upsilon}$ and lower and upper $*$-fusion frame bounds A and B, respectively. Suppose that T is an adjointable map on $\mathcal{H}$ such that $T^*T(F_j) \subseteq F_j$, for $j \in J$. Then T is surjective if and only if $\{(T(F_j), \upsilon_j)\}_{j\in J}$ is a $*$-fusion frame for $\mathcal{H}$. In this* *case,* $S_{T(F),\upsilon} := TST^{-1}$*,* $A\|(TT^*)^{-1}\|^{-\frac{1}{2}}$*, and* $B\|T\|$ *are frame operator and lower and upper* $*$*-fusion frame bounds for* $\{(T(F_j), \upsilon_j)\}_{j\in J}$*, respectively.*

We saw that the $*$-fusion frames had the elementary properties of ordinary frames. We also saw that every fusion frame is a $*$-fusion frame. The following result determines completely the relation between fusion frames and $*$-fusion frames.

Theorem 2.6. *Let* $\{(F_j, \upsilon_j)\}_{j\in J}$ *be a* $*$*-fusion frame for Hilbert* $\mathcal{A}$*-module* $\mathcal{H}$ *with pre-frame operator* $\theta_{F,\upsilon}$*. Then* $\{(F_j, \upsilon_j)\}_{j\in J}$ *is a fusion frame with real-valued bounds* $\|(\theta^*_{F,\upsilon}\theta_{F,\upsilon})^{-1}\|^{-1}$ *and* $\|\theta_{F,\upsilon}\|^2$*.*

Proof. The properties of $\theta_{F,\upsilon}$ and Lemma 2.7 [1] conclude that

$$\begin{aligned}\|(\theta^*_{F,\upsilon}\theta_{F,\upsilon})^{-1}\|^{-1}\langle f, f\rangle \leq \langle \theta_{F,\upsilon}(f), \theta_{F,\upsilon}(f)\rangle &= \sum_{j\in J} \upsilon_j^2 \langle \pi_{F_j}(f), \pi_{F_j}(f)\rangle \\ &\leq \|\theta_{F,\upsilon}\|^2 \langle f, f\rangle,\end{aligned}$$

for all $f \in \mathcal{H}$. □

We introduced lower and upper real bounds for every $*$-fusion frame and saw that $*$-fusion frames can be studied as fusion frames with different bounds.

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