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INTERNAL TOPOLOGY ON MI-GROUPS

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Abstract. An MI-group is an algebraic structure based on a generalization of the concept of a monoid that satisfies the cancellation laws and is endowed with an invertible anti-automorphism representing inversion. In this paper, a topology is defined on an MI-group G under which G is a topological MI-group. Then we will identify open, discrete and compact MI-subgroups. The connected components of the elements of G and connected MI-groups are also identified. Some features of the maximal MI-subgroups and ideals of a topological MI-group are investigated as well. Finally, some theorems about automatic continuity will be introduced.

1. Introduction

A many identities group (MI-group, in short) is a special algebraic structure in which identity like elements (called pseudoidentities) are specified and collected into a monoidal substructure. Accordingly, many algebraic structures, such as monoids of fuzzy intervals or convex bodies possessing behaviors very similar to that of a group structure, may be well described and investigated by using a new approach, which seems to be superfluous for the

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classical structures. The concept of MI-groups has recently been introduced by Holcapek and Stepnicka in the paper " MI-algebras: A new framework for arithmetics of (extensional) fuzzy numbers "[4] . The basic problem in MI-groups lacks an inverse element for each member of the MI-group. Therefore in an MI-group, the equation $ax = b$ does not necessarily have a solution. Accordingly, in many relationships and equations relating to MI-groups to find a particular member, an important MI-subgroup P_G contains symmetric elements xx^{-1}, playing a fundamental role. The main aim of the present paper is to define a particular topology on an arbitrary MI-group such that it becomes a topological MI-group. This topology is based on the algebraic properties of the MI-group, and so, as we shall see later, it has important connections with desired topological MI-groups. This topology is called the Internal MI-group topology. We will examine the properties of this topology, including the accuracy or inaccuracy of the results of topological groups. In the following section, we recall some of the basic definitions, propositions and theorems related to MI-groups. In the third section, the internal topology on MI-groups is introduced, and then by virtue of numerous theorems, some of its properties and features including open, closed and discrete MI-subgroups of an MI-group G are identified. In the fourth section, some properties of H^* containing reversible elements of an MI-group G, are specified. The fifth and sixth sections are devoted to the identification of compact subsets and connected subsets of a topological MI-group underneath the internal topology. In seven, eight and nine sections, the quotient topology, some topological properties of the maximal MI-subgroups of a topological MI-group and ideals in a topological MI-group are expressed. The final section presents some properties of automatic continuity of the homomorphisms of MI-groups.

2. Preliminaries

An MI-group G is an algebraic structure based on a generalization of the concept of a monoid that satisfies the cancellation laws and is endowed with an invertible anti-automorphism representing inversion. A fundamental role is played by pseudoidentities, i.e. elements that possess similar properties to the identity element. The most important types of such elements are the form xx^{-1}, where $x \in G$.

In this section, we first discuss the definitions and important concepts of the MI-groups.

Definition 2.1. (Definition 2.1 [5]) A triplet $(G, \star, ^{-1})$ is said to be an MI-group if it satisfies the following axioms:

(1) $(G, \star)$ is a monoid,

(2) $^{-1} : G \to G$ is an involutive anti-automorphism, i.e., $\forall x, y \in G$, it holds

$(i)(x \star y)^{-1} = y^{-1} \star x^{-1}$,

$(ii)(x^{-1})^{-1} = x$,

(3) $x \star (y \star y^{-1}) = (y \star y^{-1}) \star x$ for any $x, y \in G$,

(4) the cancellation laws hold, i.e., $\forall x, y, z \in G$,

$x \star y = x \star z \Longrightarrow y = z$ (*left cancellation law*),

$y \star x = z \star x \Longrightarrow y = z$ (*right cancellation law*).

Typically, we write $G = (G, \star, {}^{-1}, e)$ and $x \star y = xy$. Let P_G be the least submonoid of G that contains the set $\{xx^{-1} : x \in G\}$. Elements of P_G are called pseudoidentity elements, e is called an (strong) identity element and the involutive anti-automorphism ${}^{-1}$ of G will be called the inversion of G. By Lemma 2.1 of [5], we have

$$P_G = \{x_1x_1^{-1}x_2x_2^{-1}...x_nx_n^{-1} \mid x_1, x_2, ..., x_n \in G\}.$$

Moreover, $sx = xs$ for every $x \in G$ and $s \in P_G$ (Axiom (3) of definition) . Also by Lemmas 2.2 and 2.3 of [5], we get

i) $xx^{-1} = x^{-1}x, \forall x \in G$

ii) $s = s^{-1}, \forall s \in P_G$.

The recent feature shows that the elements P_G are symmetric. If $P_G = \{e\}$, then G has a group structure. It should be noted that the above definition of an MI-group and P_G can be based on the definition 2.1 from [1].

For MI-groups G and H, a mapping $f : G \to H$ is a homomorphism of MI-groups, provided that

(1) $f(x \star_G y) = f(x) \star_H f(y), \ \forall x, y \in G$,

(2) $f(e_G) = e_H$,

(3) $f(x^{-1}) = f(x)^{-1} \ \forall x \in G$.

Let H be a non- empty subset of G. The set H is said to be closed in G, if $xs \in H$ implies $x \in H$ whenever $x \in G$ and $s \in P_G$. The set

$$\overline{H}^G = \bigcap \{K \subseteq G \mid K \textit{ is closed in } G \textit{ and } H \subseteq K\}$$

is called a closure of H in G. By theorem 3.1 of [1],

$$\overline{H}^G = \{x \in G \mid \exists s \in P_G : xs \in H\}.$$

Definition 2.2. (Definition 2.8 [1]) Let $G = (G, \star, {}^{-1}, e)$ be an MI-group, and $H \subseteq G$. If $H = (H, \star, {}^{-1}, e)$ is itself an MI-group under the product and inversion of G, then H is said to be an MI-subgroup of G, which is denoted by $H \leq G$.

According to Theorem 2.4 of [5], H is an MI-subgroup of G if and only if $e \in H$ and $xy^{-1} \in H$ for each $x, y \in H$. By theorem 2.3 of [1], P_G is an MI-subgroup of G. By Lemma 2.1 of [5], P_G is also an abelian MI-subgroup of G. An MI-subgroup H of an MI-group G that contains P_G is said to be full and is denoted by $H \leq_f G$. We say H is a non-full MI-subgroup, if H is not a full MI-subgroup.

Now, like the topological groups, we have the following definition:

Definition 2.3. Suppose that G is an MI-group, whose underlying space is a topological space. Then G is called a topological MI-group if $(x, y) \to x \star y$ maps $G \times G$ onto G and $x \to x^{-1}$ maps G on G continuously.

For example, every MI-group $G = (G, \star, {}^{-1}, e)$ endowed with the discrete topology is a topological MI-group.

3. Internal topology

In this section, we define a topological structure on an arbitrary MI-group G that has a complete connection with its algebraic structure. In fact, the algebraic and topological closures of the subsets of G are the same. This topology is called the Internal MI-group topology.

Definition 3.1. (Internal topology on MI-groups) Let G be a MI-group, and U subset of G. We say that U is open in G, if $\overline{U^c}^G = U^c$, i.e. U^c is closed in G from the MI-groups point of view, where U^c is complement of U in G.

According to this definition, we get

$$U^c = \{x \in G \mid \exists\ s \in P_G\ xs \in U^c\},$$

or

$$U = \{x \in G \mid \forall s \in P_G\ xs \in U\} = \{x \in G \mid xP_G \subseteq U\}.$$

So according to this relationship, to prove the openness of a set is enough to show that $U \subseteq \{x \in G \mid xP_G \subseteq U\}$. Obviously U is open if and only if $UP_G = U$. It is clear that the family of such subsets of G, including $\emptyset$ and G, has the properties of a topology. This topology is called the Internal topology on an MI-group. It is obvious that P_G and every full MI-subgroup of G are open. The subset U of G is closed if and only if $\overline{U}^G = U$, that is, the topological closure and the MI-group closure of U are the same. Also every neighborhood of e contains P_G.

Proposition 3.2. *Let G be an topological MI-group under the internal topology and let $\mathscr{U}$ be an open basis at e . Then for every neighborhood U of e and $x \in G$, xU is also an open set of G. Also the family $\{xU\}$ and $\{Ux\}$, where x runs through all elements of G, and U runs through all elements of $\mathscr{U}$, are open bases at every $x \in G$.*

Proof. Let U be an open subset of G containing e and $x \in G$. For every $y \in xU$, there is $u \in U$ such that $y = xu$. Now for any $s \in P_G$, because $uP_G \subseteq U$, we have $us \in U$, and hence

$$ys = xus \in xU.$$

Therefore $yP_G \subseteq xU$ and so $xU = \{y \in G \mid yP_G \subseteq xU\}$. Similarly Ux is also open and so is a neighborhood of x. Also for each open subset U of G and $x \in U$, by definition of openness in the internal topology we have $xP_G \subseteq U$. Therefore, the element $V = P_G$ of $\mathscr{U}$ has been found such that $xV \subseteq U$, i.e. $\{xV\}$ is an open basis for G in x. It should be noted that each open basis $\mathscr{U}$ at e contains also P_G. Actually, because P_G is an open subset of G containing e, there is $V \in \mathscr{U}$ such that $V \subseteq P_G$. On the other hand, by definition of openness in the internal topology, for each open subset V contains e, we have $P_G \subseteq V$. Therefore $V = P_G$ and so $P_G \in \mathscr{U}$. The proof for the family $\{Ux\}$ is similar. □

Proposition 3.3. *Let G be an MI-group. Then under the Internal topology, G is a topological MI-group.*

Proof. For every $a,b \in G$ and the neighborhood abU of ab, where $U \in \mathscr{U}$, there are neighborhoods V and W of e such that $aWbV \subseteq abU$. In fact, if $V = W = P_G$ then

$$aWbV = aP_GbP_G = abP_GP_G = abP_G^2 \subseteq abP_G \subseteq abU.$$

Hence $\star$ is continuous. Also for each $a \in G$ and every neighborhood U of a^{-1}, there is $V \in \mathscr{U}$ such that $Va^{-1} \subseteq U$. We can select V such that $V = V^{-1}$. Therefor for neighborhood aV of a we have

$$(aV)^{-1} = V^{-1}a^{-1} = Va^{-1} \subseteq U$$

i.e. inversion $^{-1}$ is continuous.
Thus, each MI-group G under this topology, becomes a topological MI-group satisfying

(1) *U is open in G and $x \in G$ imply xU is open in G.*

□

Similar to the topological groups, it is easy to see:

Lemma 3.4. *Let G be a topological MI-group. Then every neighborhood U of e contains a symmetric neighborhood V, i.e. $V = V^{-1}$. Also there is a neighborhood $e \in V$ such that $V^2 \subseteq U$ and $V^{-1} \subseteq U$.*

Proof. Because of continuity of $\star$ and $^{-1}$ in (e,e) and e respectively, for every neighborhood U of e there is a neighborhood V of e such that $V^2 \subseteq U$ and $V^{-1} \subseteq U$. If $W = V \cap V^{-1}$ then W is a symmetric neighborhood of e and $W \subseteq U$. □

Remark 3.5. Based on the above discussion, since P_G is open, for each $x \in G$ xP_G is also an open subset of G. On the other hand, since every neighborhood of e contains P_G, hence P_G is the smallest open subset of G which contains e. Therefore for each $x \in G$, xP_G is the smallest and simplest open subset of G containing x.

From here on, across the text, G is a topological MI-group under the Internal topology.

Proposition 3.6. *Let G be a topological MI-group. Then for every open subset U of G, $\overline{U}^G$ is also open.*

Proof. Let U be an open subset of G. Then $U = \{x \in G \mid xP_G \subseteq U\}$. Let $x \in \overline{U}^G$. Therefore there is $s \in P_G$ such that $xs \in U$ and so $xsP_G \subseteq U$. Hence for each $t \in P_G$, we have $xst \in U$, or $xts \in U$. Hence for each $t \in P_G$, we have $xt \in \overline{U}^G$ or $xP_G \subseteq \overline{U}^G$. Therefore $\overline{U}^G = \{x \in G \mid xP_G \subseteq \overline{U}^G\}$, i.e. $\overline{U}^G$ is open. □

Proposition 3.7. *Let G be a topological MI-group such that P_G is closed ($\overline{P_G}^G = P_G$). Then for each $x \notin P_G$, $xP_G \cap P_G = \emptyset$.*

Proof. Let $xP_G \cap P_G$ contains y. Then there is $s \in P_G$ such that $y = xs$. Then since $y \in P_G$, $x \in \overline{P_G}^G = P_G$, which is a contradiction. □

Corollary 3.8. *Every $x \notin P_G$ and $y \in P_G$ have disjoint neighborhoods xP_G and P_G, i.e. they are separated from each other.*

Proposition 3.9. *Let G be a topological MI-group. Then for every $x \in G$*

$$e \in \overline{\{x\}}^G \Longleftrightarrow x \in P_G$$

Proof. Let $e \in \overline{\{x\}}^G$. Hence there is $s \in P_G$ such that $es \in \{x\}$ or $s = x$, i.e. $x \in P_G$. Now if $x \in P_G$, then $ex = x \in \{x\}$ implies that $e \in \overline{\{x\}}^G$. □

Corollary 3.10. *For each $x \in P_G$, we have*

$$\overline{\{x\}}^G = \{x\} \Longrightarrow x = e$$

Therefore, among the members of P_G, only for e, $\{e\}$ can be closed.

Example 3.11. Let $G = \{[a,b] \mid a,b \in \mathbb{R}, a \le b\}$ be the set of all closed real intervals. Then by example 2.2 of [1], we know $(G, +, -, [0,0])$ under algebraic actions

$$[a,b] + [c,d] = [a+c, b+d],$$
$$-[a,b] = [-b,-a],$$

is an additive abelian MI-group. Obviously, $P_G = \{[-x,x] \mid x \ge 0\}$.

Example 3.12. The subset $G^* = \{(a,b) \in \mathbb{R}^2 \mid a,b \in \mathrm{R}, \mathrm{a} \le \mathrm{b}\}$ of $\mathbb{R}^2$ under actions

$$(a,b) + (c,d) = (a+c, b+d),$$
$$-(a,b) = (-b,-a),$$

is also an additive abelian MI-group of pairs of real numbers. It is easily seen that the mapping

$$f : G^* \to G,\ f(a,b) = [a,b]$$

is an isomorphism of MI-groups G^* and G. Now as a subspace of $\mathbb{R}^2$ with its relative euclidean topology, G^* is a topological MI-group. Indeed the actions $+$ and $-$, are obviously continuous. Therefore under the topology on G induced by f, G is a topological MI-group. In fact, under this mapping, each topology of one of the two algebraic structures induces a topology on another.

Remark 3.13. For every $[a,b] \in G$, suppose that $[c,d] \in \overline{\{[a,b]\}}^G$. Then there exists $[-x,x] \in P_G$ such that $[c,d] + [-x,x] \in \{[a,b]\}$ or $[c-x, d+x] = [a,b]$. Hence we conclude that $c = a+x, d = b-x$, and hence $a+b = c+d$, i.e. $[a,b]$ and $[c,d]$ have the same center and

$$\overline{\{[a,b]\}}^G = \{[a+x, b-x] :\ 0 \le x \le \frac{b-a}{2}\}.$$

Therefore single-member sets are not closed in MI-group G of real intervals, unless $\{[a,a]\}$.

Example 3.14. Let $G^+ = \{[a,b] \mid a,b \in \mathbb{R}^+, a \le b\}$ be the set of all closed real intervals of positive real numbers. Similar to Example 3.11, it is easy to see that G^+ under algebraic actions . and $^{-1}$ defined as follows is an abelian multiplicative MI-group:

$$[a,b].[c,d] = [a.c, b.d],$$
$$[a,b]^{-1} = [1/b, 1/a].$$

We will write in this case $G^+ = (G^+, ., ^{-1}, [1,1])$, where $[1,1]$ is the identity element of G^+.

Let $Bd(U) = \overline{U} \cap \overline{U^c}$ be topological boundary of U, where $\overline{U} = \overline{U}^G$.

Theorem 3.15. *Let G be a topological MI-group with identity e and U be an open subset of G such that $e \notin U$. If $e \notin Bd(U)$, then $P_G \cap U = \emptyset$.*

Proof. By assumption, since $e \in U^c$ and $e \notin Bd(U)$, hence e is an interior point of U^c and so there is a neighborhood V of e such that $V \subseteq U^c$. Since $P_G \subseteq V$, we have $P_G \subseteq U^c$. Therefore $P_G \cap U = \emptyset$. □

As stated in the introduction, most elements of an MI-group have no inverse element. In the rest of our study on MI-groups, we examine some of the characteristics of the set of reversible elements of an MI-group. First, we start with a definition.

Definition 3.16. Let G be a topological MI-group. we define

$$H^* = \{x \in G \mid xx^{-1} = e\}.$$

It's easy to prove that H^* is a MI-subgroup of G and in fact have a group structure.

Proposition 3.17. *Let G be a MI-group such that $\overline{\{e\}}^G = \{e\}$. Then H^* is also closed, i.e. $\overline{H^*}^G = H^*$.*

Proof. Suppose that $x \in \overline{H^*}^G$. Then $xs \in H^*$ for a suitable $s \in P_G$. Hence $(xs)(xs)^{-1} = e$ or $xss^{-1}x^{-1} = e$ or $xx^{-1}ss^{-1} = e$. Then since $ss^{-1} \in P_G$ and $xx^{-1}ss^{-1} = e \in \{e\}$, we have $xx^{-1} \in \overline{\{e\}}^G = \{e\}$, i.e. $xx^{-1} = e$ or $x \in H^*$. □

Lemma 3.18. *Let G be a MI-group such that $\overline{\{e\}}^G = \{e\}$. Then $P_G \cap H^* = \{e\}$.*

Proof. Let $x \in P_G \cap H^*$. Hence $xx^{-1} = e \in \{e\}$. Since $x \in P_G$, therefore $x^{-1} \in P_G$ and so $x \in \overline{\{e\}}^G = \{e\}$, i.e. $x = e$. □

Proposition 3.19. *Let G be a topological MI-group such that $\overline{\{e\}}^G = \{e\}$. Then H^* is a discrete MI-subgroup of G.*

Proof. Suppose that $x \in H^*$. Then for each $y \in xP_G \cap H^*$, there is $s \in P_G$ such that $y = xs$. Since $y \in H^*$ and $xx^{-1} = e$, we have $(xs)(xs)^{-1} = e$ or $xx^{-1}ss^{-1} = e$, i.e $ss^{-1} = e$. But by lemma 3.18, $s = e$ and so $y = x$. Therefore $xP_G \cap H^* = \{x\}$, i.e x is an isolated point of H^*. □

Example 3.20. Let $(G, +, -, [0, 0])$ be the additive abelian MI-group of real intervals (Example 3.11). Then we have

$$\begin{aligned} H^* &= \{[a,b] : [a,b] + [-b,-a] = [0,0]\} \\ &= \{[a,b] : [a-b, b-a] = [0,0]\} \\ &= \{[a,a] : a \in \mathbb{R}\} \end{aligned}$$

Similarly, for example 3.14, the same is true.

Now, we list a number of theorems and corollaries from open, closed and discrete MI-subgroups of a topological MI-group. At first, we start with a study of open MI-subgroups.

Theorem 3.21. *Let G be a topological MI-group and H be a canonical MI-subgroup of G. Then H is an open MI-subgroup of G if and only if it is full.*

Proof. If H is a full MI-subgroup of G, then $P_G \subseteq H$ and so for each $x \in H$ we have $xP_G \subseteq H$. Therefore $H = \{x : xP_G \subseteq H\}$, i.e. H is open. Conversely, suppose that H is an open MI-subgroup of G. Since $e \in H$, hence $P_G = eP_G \subseteq H$ and so H is a full MI-subgroup of G. □

Theorem 3.22. *A MI-subgroup H of topological MI-group G is open if and only if e is an interior point of it.*

Proof. Obviously, if H is open , then e is an interior point of it. Conversely, let e be an interior point of H. Hence there is a neighborhood U of e such that $U \subseteq H$. Since $P_G \subseteq U$, hence $P_G \subseteq H$, i.e. H is full. By the previous theorem, H is open. □

It should be noted that a symmetric neighborhood of e is a neighborhood satisfying $V = V^{-1}$.

Theorem 3.23. *Let G be a topological MI-group. Then for every symmetric neighborhood U of e, $H = \overset{\infty}{\underset{n=1}{\cup}} U^n$ is an open MI-subgroup of G.*

Proof. The proof is clear. □

Remark 3.24. It should be noted that unlike topological groups, in topological MI-groups, each open MI-subgroup is not necessarily closed. Indeed let $(G, +, -, [0, 0])$ be the additive abelian MI-group of real intervals and

$$H = \bigcup_{[m,n]\in A} ([m, n] + P_G) \cup P_G,$$

where $A = \{[m, n] \in G : m < n,\ m, n \in \mathbb{Z}\}$. Obviously, H is an open MI-subgroup of G, but not closed. In fact, for every $m, n \in \mathbb{Z}$ such that $m < n$, we have

$$\overline{[m, n] + P_G}^G = [\frac{m+n}{2}, \frac{m+n}{2}] + P_G.$$

But H does not have such a subsets of G.

A discrete MI-subgroup of G is a MI-subgroup H that is discrete under the space topology, i.e. each singleton is an open set in H and so all subsets of it are open. A point x is called an isolated point of a subset S (in a topological space X) if x is an element of S but there exists a neighborhood U of x which does not contain any other points of S, i.e. $U \cap S = \{x\}$. This is equivalent to saying that the singleton $\{x\}$ is an open set in the topological space.

Theorem 3.25. *Let G be a topological MI-group such that $P_G \neq \{e\}$,(i.e. G does not have a group structure) and H be a full MI-subgroup of G. Then there is no isolated point for H.*

Proof. Let $x \in H$. For every neighborhood U of x, we have $xP_G \subseteq U$. On the other hand, since H is full, it is an open subset of G and $P_G \subseteq H$ and so $xP_G \subseteq H$. Therefore $xP_G \subseteq U \cap H$. By assumption since $P_G \neq \{e\}$, we have $xP_G \neq \{x\}$. Hence $U \cap H \neq \{x\}$. That is, x is not an isolated point of H. □

Theorem 3.26. *Let H be a non-full MI-subgroup of topological MI-group G . If x is an isolated point of H, then $x \in H^*$.*

Proof. Suppose that $x \in H$ is an isolated point. Thus there is a neighborhood xU of x such that $xU \cap H = \{x\}$, where U is neighborhood of e. Hence $xP_G \cap H = \{x\}$ since $P_G \subseteq U$. But since $xx^{-1} \in P_G \cap H$, we have

$$x(xx^{-1}) \in xP_G \cap H = \{x\}.$$

Therefore $x(xx^{-1}) = x$, and finally by left cancellation law, we obtain that $xx^{-1} = e$, i.e. $x \in H^*$. □

Corollary 3.27. *Let H be a discrete non-full MI-subgroup of topological MI-group G. Then $H \leq H^*$. Therefore H^* is the largest discrete MI-subgroup of G. Every discrete MI-subgroup of G, also has a group structure.*

Theorem 3.28. *Let H be a non-full MI-subgroup of topological MI-group G. If e is an isolated point of H, then $H \leq H^*$.*

Proof. Since e is an isolated point of H, there is a neighborhood of e such that $U \cap H = \{e\}$ and so $P_G \cap H = \{e\}$. On the other hand, since H is an MI-subgroup, for each $x \in H$ we have $xx^{-1} \in H$ and so

$$\{xx^{-1} : x \in H\} \subseteq P_G \cap H = \{e\}.$$

Therefore for every $x \in H$, we get $xx^{-1} = e$, i.e. $x \in H^*$. □

Theorem 3.29. *Let H be a non-full MI-subgroup of topological MI-group G. Suppose that $\{e\}$ is closed. Then H is discrete if and only if e is an isolated point of H .*

Proof. If H is discrete, then by definition of a discrete MI-subgroup, all of its points including e are isolated. Conversely if e is an isolated point of H, then by theorem 3.28, $H \leq H^*$ and so by Proposition 3.19, H is discrete. □

Theorem 3.30. *Let H be a non-full MI-subgroup of topological MI-group G. Suppose that $\{e\}$ is closed. If H has an isolated point, then it is discrete.*

Proof. By assumption, there is $x \in H$ such that x is an isolated point of it. Hence $xP_G \cap H = \{x\}$. Let $s \in P_G \cap H$. Then $xs \in xP_G$. Since $x \in H$, $xs \in H$ and so $xs \in xP_G \cap H = \{x\}$. Therefore $xs = x$ and by cancellation law $s = e$. Hence $P_G \cap H = \{e\}$, i.e. e is an isolated point of H. By the previous theorem, H is discrete. □

According to Theorems 3.29 and 3.30, we have

Theorem 3.31. *Let H be a non-full MI-subgroup of topological MI-group G. Suppose that $\{e\}$ is closed . Then H is discrete if and only if it has an isolated point .*

Theorem 3.32. *Let G be a topological MI-group such that $\{e\}$ is closed. Then every discrete non-full MI-subgroup H of G is closed.*

Proof. Suppose that $x \in \overline{H}^G$. By corollary 3.27, $H \subseteq H^*$ and so $\overline{H}^G \subseteq \overline{H^*}^G$. Since by proposition 3.17, H^* is closed, hence $x \in H^*$, i.e. $xx^{-1} = e$. On the other hand there is $s \in P_G$ such that $xs \in H$. Therefore $xs \in H^*$ and

$$(xs)(xs)^{-1} = e \text{ or } xss^{-1}x^{-1} = e \text{ or } ss^{-1} = e.$$

Hence $ss^{-1} = e \in \{e\}$. Since $s^{-1} \in P_G$, hence $s \in \overline{\{e\}}^G = \{e\}$, i.e. $s = e$ or $x \in H$. □

4. Some properties of H^*

Let $(G, +, -, [0, 0])$ be the additive abelian MI-group of real intervals (Example 3.11). By example 3.20, for each $a \in \mathbb{R}$, we have $[a, a] \in H^*$. On the other hand, it's easy to see $[a, a]+P_G$ is a closed subset of G and

$$\overline{[a, b] + P_G}^G = [\frac{a + b}{2}, \frac{a + b}{2}] + P_G. \quad (2)$$

Therefore $[a, b] + P_G$ is closed if and only if $a = b$, i.e. $[a, b] \in H^*$. In generally, we also have:

Theorem 4.1. *Let G be a topological MI-group such that P_G is closed. If $x \in H^*$ then xP_G is closed.*

Proof. Suppose that $x \in H^*$. Hence $xx^{-1} = e$. Let $y \in \overline{xP_G}^G$. Then $\overline{yP_G}^G = \overline{xP_G}^G$ because $y \in \overline{yP_G}^G$ and two left cosets are either equal or disjoint. Therefore $x^{-1}y \in \overline{P_G}^G = P_G$ by Corollary 3.6 of [1]. Therefore $y \in xP_G$. □

Remark 4.2. If P_G is closed, the above theorem for any desired topology on G is also correct. Indeed let x_α, $\alpha \in D$, be a net in xP_G that converges to $x_0 \in G$. Hence there is a net $s_\alpha \in P_G$ such that $x_\alpha = xs_\alpha$. Since $x \in H^*$, we have $s_\alpha \to x^{-1}x_0$. Therefore since P_G is closed, $x^{-1}x_0 \in P_G$ and so $x_0 \in xP_G$.

To verify the converse of Theorem 4.1, we need certain conditions on MI-groups. Therefore, we provide the following definition that expresses the property of the MI-groups.

Definition 4.3. A topological MI-Group G is said to have the property I, whenever for any $x \in G$, $\overline{xP_G}^G \cap H^* \neq \emptyset$.

By equation (2), the MI-group of real intervals $(G, +, -, [0, 0])$ clearly has the property I.

Theorem 4.4. *Let G be a topological MI-group such that $\{e\}$ is closed and also have the property I. For every $x \in G$, if xP_G is closed, then $x \in H^*$.*

Proof. Let $x \in G$ be arbitrary such that xP_G is a closed subset of G, i.e. $\overline{xP_G}^G = xP_G$. Since G has the property I, we have $xP_G \cap H^* = \overline{xP_G}^G \cap H^* \neq \emptyset$. Hence there is $y \in xP_G \cap H^*$, so that $y = xs$, where $s \in P_G$. Since $y \in H^*$, we have $(xs)(xs)^{-1} = e$ or $xx^{-1}ss^{-1} = e \in \{e\}$. Thus $xx^{-1} \in \overline{\{e\}}^G = \{e\}$, i.e. $xx^{-1} = e$. □

In this way, the following theorem is proved.

Theorem 4.5. *Let G be a topological MI-group such that $\{e\}$ and P_G are closed. Also suppose that G have the property I. Then xP_G is closed subset of G if and only if $x \in H^*$.*

Definition 4.6. A topological MI-Group G is said to have the property II, if for each $x \in G$, there is $y \in H^*$ such that $\overline{xP_G}^G = yP_G$.

By equation (2), the MI-group of real intervals $(G, +, -, [0, 0])$ clearly has the property II, since for each $[a, b] \in G$, $[\frac{a+b}{2}, \frac{a+b}{2}] \in H^*$. It is easily seen that the properties I and II on every MI-group G are equivalent. Therefore, by replacing the property I with II, the theorem 4.5 remains true.

Definition 4.7. Let G be a topological MI-group. we define

$$H^{**} = \{g \in G : gG = G\}.$$

Obviously $e \in H^{**}$ and so $H^{**} \neq \emptyset$.

Proposition 4.8. *Let G be a topological MI-group such that $\{e\}$ is closed. Then $H^{**} = H^*$.*

Proof. Let $g \in H^{**}$. Then $gG = G$ and so there is $g_1 \in G$ such that $gg_1 = e$. Therefore we will have $g_1g = g_1eg = g_1(gg_1)g = (g_1g)(g_1g)$ and so by law cancellation we have $g_1g = e$. Hence $g_1G = g_1gG = eG = G$, i.e. $g_1 \in H^{**}$. On the other hand since $g^{-1}g_1^{-1} = e$, hence for every $u \in G$, we have $u = g^{-1}g_1^{-1}u \in g^{-1}G$ and so $G \subseteq g^{-1}G$, i.e. $g^{-1}G = G$ or $g^{-1} \in H^{**}$. According to relationship $g^{-1}G = G$ there is $x \in G$ such that $g^{-1}x = e$ and also according to relationship $gG = G$, there is $y \in G$ so that $gy = x$. Hence $e = g^{-1}x = g^{-1}gy$. Since $g^{-1}g \in P_G$, hence $y \in \overline{\{e\}}^G = \{e\}$, i.e. $y = e$ and $g = x$. Therefore $g^{-1}g = e$ or $g \in H^*$. Conversely, if $g \in H^*$ then $g^{-1}g = e$. Hence for each $u \in G$, $u = g^{-1}gu \in gG$ and so $G \subseteq gG$ or $gG = G$, i.e. $g \in H^{**}$. Therefore $H^{**} = H^*$ and obviously H^{**} is also a MI-subgroup of G. □

5. Compact subsets of G

Let G is a topological MI-group under the internal topology. In this section, we identify the compact subsets of a topological MI-group under the internal topology.

Theorem 5.1. *Let G be a topological MI-group. Then P_G is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of P_G. Hence there is $\alpha_0 \in I$ such that $e \in U_{\alpha_0}$. Hence by definition of the open subsets of G, $P_G = eP_G \subseteq U_{\alpha_0}$. □

Remark 5.2. By the preceding theorem, for each $a \in G$, since the mapping $x \to ax$ is continuous, aP_G is compact. On the other hand, aP_G is a neighborhood of a and so for every $a \in G$, aP_G is a compact neighborhood of it. Therefore for each $a \in G$, there is a compact set aP_G containing the neighborhood aP_G of a, i.e. G is a locally compact space.

Theorem 5.3. *Let H be a compact full MI-subgroup of a topological MI-group G. If $\{e\}$ is closed, then there are $x_1, x_2, ..., x_n \in H$ such that*

$$H = P_G \cup x_1P_G \cup x_2P_G \cup ... \cup x_nP_G.$$

Proof. Since for every $x \in G$, xP_G is open subset of G, Hence $\{xP_G\}_{x\in H}$ is an open cover of H. Therefore, because of the compctness of H there are $x_1, x_2, ..., x_m \in H$ such that $H \subseteq \overset{m}{\underset{i=1}{\cup}} x_iP_G$. Since $e \in H$, there is x_i such that $e \in x_iP_G$ and so there exists $s \in P_G$ so that $e = x_is$. Thus $x_i \in \overline{\{e\}}^G = \{e\}$, i.e. $x_i = e$. Therefore, by renaming, we see that

$$H \subseteq P_G \cup x_1P_G \cup x_2P_G \cup ... \cup x_nP_G.$$

On the other hand since H is full, hence is open and so for every $x_i \in H$, $x_iP_G \subseteq H$. Therefore $H = P_G \cup x_1P_G \cup x_2P_G \cup ... \cup x_nP_G$. □

We need to remind that by definition 4.1 of [1], a full MI-subgroup H of a MI-group G is said to be a normal full MI-subgroup of G, if for each $x \in G$, $\overline{xH}^G = \overline{Hx}^G$. We write $H \lhd G$ if H is normal in G.

Proposition 5.4. *Let G be a topological MI-group such that P_G is closed. Then every neighborhood U of e contains a normal MI-subgroup of G that is open, closed and compact.*

Proof. By theorem 4.3 of [1], P_G is normal in G. Also according to Theorems 3.21 and 5.1 and Remark 3.5, P_G is compact and open. Because each neighborhood U of e contains P_G, the result is clear. □

Lemma 5.5. *Let G be a topological MI-group. Then for every subset F of G*

$$x \in \overline{F}^G \Longleftrightarrow xP_G \cap F \neq \emptyset.$$

Proof. Let $x \in \overline{F}^G$. Then there is $s \in P_G$ so that $xs \in F$. Since $xs \in xP_G$, hence $xs \in xP_G \cap F$. The proof of the opposite direction is similar. □

Theorem 5.6. *Let G be a topological MI-group and U be a compact subset of G. Then every subset F of U is compact.*

Proof. Since every open subset V of G is in the form $V = \underset{x\in V}{\cup} xP_G$, therefore, it is sufficient only to consider the open covers of each subset F of G in the form $\{xP_G\}_{x\in A}$, where $A \subseteq G$. If $x \notin \overline{F}^G$, then by the previous lemma $xP_G \cap F = \emptyset$. Hence $A = \overline{F}^G$. Now the family of open subsets $\{xP_G\}_{x\in\overline{F}^G} \cup \{yP_G\}_{y\notin\overline{F}^G}$ is an open cover of U and so there are $x_1, x_2, ..., x_n \in \overline{F}^G$ and $y_1, y_2, ..., y_m \notin \overline{F}^G$ such that

$$U \subseteq (\overset{n}{\underset{i=1}{\cup}} x_iP_G) \cup (\overset{m}{\underset{j=1}{\cup}} y_jP_G).$$

Since for each $y_j \notin \overline{F}^G$, $y_jP_G \cap F = \emptyset$, so obviously $F \subseteq \overset{n}{\underset{i=1}{\cup}} x_iP_G$, i.e. F is a compact subset of U. □

Proposition 5.7. *Let G be a compact topological MI-group such that $\{e\}$ is closed. Then H^* is a finite set.*

Proof. By Theorem 5.3, there are $x_1, x_2, ..., x_n \in G$ such that

$$G = P_G \cup x_1P_G \cup x_2P_G \cup ... \cup x_nP_G.$$

Now for every $g \in H^*$, by Proposition 6, $g \in H^{**}$ and so $gG = G$. Therefore

$$gP_G \cup gx_1P_G \cup gx_2P_G \cup ... \cup gx_nP_G = G = P_G \cup x_1P_G \cup x_2P_G \cup ... \cup x_nP_G.$$

Hence for $e \in G$, there is $x_i \in G$ such that $e \in gx_iP_G$ or $e \in gP_G$. If $e \in gx_iP_G$ then $gx_is = e$ for one $s \in P_G$. Thus $gx_i \in \overline{\{e\}}^G = \{e\}$ or $gx_i = e$, i.e. $x_i = g^{-1}$. By using g^{-1} instead of g, we get $g = x_i$. If $e \in gP_G$, then there exist $s \in P_G$ such that $e = gs$ and hence $g \in \overline{\{e\}}^G = \{e\}$, i.e. $g = e$. Therefore $H^* \subseteq \{e, x_1, x_2, ..., x_n\}$. □

6. Connectedness in the internal topology

Let G be a topological MI-group under the internal topology. In this part, we take up some properties of topological MI-groups that depend upon connectedness or disconnectedness of an MI-group considered as a topological MI-group. Particularly the connected component of the identity element e of G is specified.

Theorem 6.1. *Let G be a topological MI-group. Then P_G is a connected subset of G.*

Proof. Suppose that $P_G = U \cup V$, where U, V are nonempty open subsets of G such that $U \cap V = \emptyset$. Since $e \in P_G$, hence $e \in U$ or $e \in V$. By definition of open sets in Internal topology, $P_G \subseteq U$ or $P_G \subseteq V$. Therefore $V = \emptyset$ or $U = \emptyset$ and so there does not exist a separation of P_G. □

Remark 6.2. If C is the connected component of the identity e, then by Theorem 18, we have $P_G \subseteq C$ and so C is open subset of G. Indeed for every $x \in C$, xP_G is the image of P_G under the continuous map $t \to xt$ and so is connected subset of G containing x. Therefore $xP_G \subseteq C$ and so $C = \{x : xP_G \subseteq C\}$, i.e. C is open .

Theorem 6.3. *Let G be a topological MI-group and let C be the connected component of the identity e in G. Then C is a closed, open and normal MI-subgroup of G.*

Proof. Since inversion is a homeomorphism of G, C^{-1} is a connected subset of G containing e, and hence $C^{-1} \subseteq C$. Therefore, if $a \in C$, then $a^{-1} \in C$. On the other hand, for each $a \in C$, Due to continuity mapping $x \to ax$, aC is a connected set containing a, and so $aC \subseteq C$. (connected components are the eqivalence classes and so C is also the eqivalence class of every

$a \in C$) Therefore $C^2 \subseteq C$ and so by Remark 6.2, C is a full MI-subgroup of G. Since $\overline{C}^G$ is also a connected subset of G containing e, we have $\overline{C}^G \subseteq C$, and so $\overline{C}^G = C$, i.e. C is closed subset of G. Eventually for every $a \in C$, depending on continuity mapping $x \to axa^{-1}$, aCa^{-1} is a connected subset of G containing aa^{-1}. Since $aa^{-1} \in P_G \subseteq C$, we will have $aCa^{-1} \subseteq C$ and so $\overline{aCa^{-1}}^G \subseteq \overline{C}^G$. Therefore by theorem 4.2 of [1], C is a normal full MI-subgroup of G. □

Theorem 6.4. *Let G be a topological MI-group and let C be the connected component of the identity e in G. Then $C = \overline{P_G}^G$.*

Proof. By Remark 6.2 and Theorem 6.3, $P_G \subseteq C$ and so $\overline{P_G}^G \subseteq \overline{C}^G = C$. On the other hand, since $\overline{P_G}^G$ is closed, hence $C - \overline{P_G}^G = C \cap (\overline{P_G}^G)^c$ is an open subset of G. Therefore since $\overline{P_G}^G$ is also an open subset of G (By Proposition 3.6), We have to have $C - \overline{P_G}^G = \emptyset$ or $C = \overline{P_G}^G$, Otherwise $C = \overline{P_G}^G \cup (C - \overline{P_G}^G)$ will be a separation of C. □

Remark 6.5. Let $(G, +, -, [0, 0])$ be the additive abelian MI-group of real intervals (Example 3.11). It is easy to see P_G is closed and hence $C = \overline{P_G}^G = P_G$. Therefore $G = P_G \cup (P_G)^c$, i.e. G is not connected space.

Theorem 6.6. *Let G be a topological MI-group and let C be the connected component of the identity e in G. Then $C = \bigcap_{H \in \mathscr{C}} \overline{H}^G$, where $\mathscr{C}$ is the family of the full MI-subgroups H of G.*

Proof. For every $H \in \mathscr{C}$, since H is full, hence $P_G \subseteq H$. Therefore $C = \overline{P_G}^G \subseteq \overline{H}^G$ and so $C \subseteq \bigcap_{H \in \mathscr{C}} \overline{H}^G$. On the other hand P_G is also a full MI-subgroup of G. Hence $\bigcap_{H \in \mathscr{C}} \overline{H}^G \subseteq \overline{P_G}^G = C$. □

Theorem 6.7. *Let G be a connected topological MI-group. Then $G = \overline{P_G}^G$.*

Proof. Obviously, G is the connected component of e. Therefore by theorem 6.6, we will have $G = C = \overline{P_G}^G$. □

Theorem 6.8. *Let G be a connected topological MI-group. Then every full MI-subgroup H of G is densed.*

Proof. Let H be a full MI-subgroup of G. By recent theorems, we have

$$G = C = \bigcap_{F \in \mathscr{C}} \overline{F}^G \subseteq \overline{H}^G,$$

hence $G = \overline{H}^G$ and H in G is densed. □

Remark 6.9. In every connected topological MI-group G, there is no proper closed and full MI-subgroup. In fact if H is a closed and full MI-subgroup of G, then by the previous theorem, $G = \overline{H}^G = H$.

According to the above theorems we have:

Theorem 6.10. *Let G be a topological MI-group such that P_G is closed. Then the following are equivalent:*

(i) G is connected.

(ii) G has no proper open MI-subgroups.

(iii) G has no proper closed full MI-subgroups.

(iv) for every neighborhood of e, $\bigcup_{n=1}^{\infty} U^n = G$.

Remark 6.11. By the previous theorem, a connected topological MI-group is the smallest possible topological MI-group, which is merely containing P_G. Therefore every connected topological MI-group is commutative.

7. The Quotient topology

Let G be a topological MI-group, and let H be a full MI-subgroup of G. By definition 3.3 and theorem 3.5 of [1], we know the equivalence class of $x \in G$ under right(resp. left) congruence module H is the closure of the right coset Hx(resp. the left coset xH) in G, i.e. $\overline{xH}^G(\overline{Hx}^G)$. Therefore, let $G/H = \{\overline{xH}^G : x \in G\}$, and φ be the natural mapping $x \to \overline{xH}^G$ of G onto G/H. Define a topology for G/H by requiring that a subset $\{\overline{xH}^G : x \in X\}$ of G/H, is open if and only if $\varphi^{-1}\{\overline{xH}^G : x \in X\}$ is open in G. It is easy to see that

$$\varphi^{-1}\{\overline{xH}^G : x \in X\} = \overline{XH}^G.$$

Therefore $\{\overline{xH}^G : x \in X\}$ is open in G/H if and only if $\overline{XH}^G$ is open in G. The topological space G/H is called the quoteint space of G by H.

Lemma 7.1. *Let G be a topological MI-group and let H be a full MI-subgroup of G. Then for every $X \subseteq G$, $\overline{XH}^G$ is open.*

Proof. Since H is a full MI-subgroup of G, by theorem 3.21, it is open and so for each $X \subseteq G$, XH is also an open subset of G. In fact $XH = \bigcup_{x \in X} xH$ and every xH is open in G. Therefore by proposition 3.6, $\overline{XH}^G$ is open. □

Theorem 7.2. *Let G be a topological MI-group, and let H be a full MI-subgroup of G. Then G/H is a discrete space.*

Proof. For every $X \subseteq G$, by lemma 7.1 since $\overline{XH}^G$ is an open subset of G, hence $\{\overline{xH}^G : x \in X\}$ is open in G/H. Therefore for each $x \in G$, $\{\overline{xH}^G\}$ is open in G/H and so every subset of G/H is open. □

Remark 7.3. By the previous theorem, for every $x \in G$, the connected component of $\overline{xH}^G$ is $\{\overline{xH}^G\}$. Therefore G/H is a totally disconnected hausdorff space. If H is normal in G, then G/H is also a discrete topological MI-group. Specifically for the connected component C of e in G, G/C is a totally disconnected Hausdorff MI-group.

Let H be a normal full MI-subgroup of G. By theorem 4.8 of [1], G/H is an MI-group under the binary operation $\star$ and the inversion $^{-1}$, defined by

$$\overline{xH}^G \star \overline{yH}^G = \overline{xyH}^G,\ (\overline{xH}^G)^{-1} = \overline{x^{-1}H}^G.$$

The MI-group G/H is called the quotient MI-group of G by H.

Theorem 7.4. *Let G be a topological MI-group and H be a full MI-subgroup of G. For each $a \in G$, Suppose that, the mapping $\psi_a : G/H \to G/H$ is defined as follows:*

$$\psi_a(\overline{xH}^G) = \overline{axH}^G,\ \forall\ \overline{xH}^G \in G/H.$$

Then ψ_a is a homeomorphism of G/H. Therefore G/H is a homogeneous space.

Proof. It should be noted that ψ_a is well defined. Indeed if $\overline{xH}^G = \overline{yH}^G$, then $x^{-1}y \in \overline{H}^G$ and hence there is $s \in P_G$ such that $x^{-1}ys \in H$. So, by multiplying $a^{-1}a$, we will see that $a^{-1}ax^{-1}ys \in a^{-1}aH$ or $(ax)^{-1}(ay)s \in a^{-1}aH$. Since H is full, $P_G \subseteq H$ and hence $(ax)^{-1}(ay)s \in H$, so that $(ax)^{-1}(ay) \in \overline{H}^G$, i.e. $\overline{axH}^G = \overline{ayH}^G$. Also ψ_a is one-to-one. In fact if $\overline{axH}^G = \overline{ayH}^G$, then $(ax)^{-1}(ay) \in \overline{H}^G$ or $x^{-1}ya^{-1}a \in \overline{H}^G$. Since $\overline{H}^G$ is closed in MI-group G ($\overline{\overline{H}^G}^G = \overline{H}^G$) and $a^{-1}a \in P_G$, $x^{-1}y \in \overline{H}^G$, i.e. $\overline{xH}^G = \overline{yH}^G$. On the other hand, for each $\overline{yH}^G \in G/H$, by lemma 4.1 of [1] and that, for each $s \in P_G$, $\overline{sH}^G = \overline{H}^G$, we have

$$\psi_a(\overline{a^{-1}yH}^G) = \overline{aa^{-1}yH}^G = \overline{yaa^{-1}H}^G = \overline{y\overline{aa^{-1}H}^G}^G = \overline{y\overline{H}^G}^G = \overline{yH}^G.$$

Therefore ψ_a is onto. It is also easy to see $(\psi_a)^{-1} = \psi_{a^{-1}}$. Therefore we need show that ψ_a is an open mapping. Let $\{\overline{uH}^G :\ u \in U\}$ be an open in G/H, where U is an open subset of G. Then

$$\psi_a(\{\overline{uH}^G :\ u \in U\}) = \{\overline{auH}^G :\ u \in U\} = \{\overline{vH}^G :\ v \in aU\}.$$

But this is an open subset of G/H, because aU is open in G. □

Finally we show that the quotient MI-group G/H under the quotient topology is a topological MI-group:

Theorem 7.5. *Let G be a topological MI-group and H be a normal full MI-subgroup of G. Then the quotient MI-group G/H under the quotient topology is a topological MI-group. Also the natural mapping φ of G onto G/H is an open , continuous homomorphism of G onto G/H.*

Proof. Let $U^* = \{\overline{uH}^G : u \in U\}$ be an arbitrary neighborhood of the identity $\overline{H}^G$ of G/H, where U is a neighborhood of e. By Lemma 3.4, there is a neighborhood V of e in G such that $V^2 \subseteq U$. Clearly the subset $V^* = \{\overline{vH}^G : v \in V\}$ of G/H is open in G/H and moreover,

$$V^{*2} = \{\overline{v_1 v_2 H}^G : v_1, v_2 \in V\} = \{\overline{yH}^G : y \in V^2\} \subseteq \{\overline{uH}^G : u \in U\} = U^*.$$

Similarly there is a neighborhood V of e such that $V^{-1} \subseteq U$ and so

$$V^{*-1} = \{\overline{vH}^{G-1} : v \in V\} = \{\overline{v^{-1}H}^G : v \in V\} \subseteq \{\overline{uH}^G : u \in U\} = U^*.$$

Therefore the mappings $(\overline{xH}^G, \overline{yH}^G) \to \overline{xyH}^G$ and $\overline{xH}^G \to \overline{x^{-1}H}^G$ are continuous at $\overline{H}^G$ and hence by Theorem 7.4, are continuous in every point of G/H.

We can also see the family $\mathscr{U}^*$ of all neighborhoods $U^* = \{\overline{uH}^G : u \in U\}$ of the identity $\overline{H}^G$ of G/H, where U is a neighborhood of e, satisfies conditions (4.5.i)-(4.5.v) of [2] and so $\mathscr{U}^*$ is an open basis at $\overline{H}^G$. Therefore by Theorem (4.5) of [2], the family of sets $\{\overline{aH}^G U^*\}$, where U^* runs through $\mathscr{U}^*$ and $\overline{aH}^G$ runs through G/H, is an open basis for the quotient topology on G/H. Properties (4.5.i) and (4.5.ii) were proved in the previous paragraph. To verify other statements of Theorem (4.5) of [2] for G/H, let $\overline{u_0 H}^G$ be an arbitrary element of U^*, where $u_0 \in U$. Hence there is a neighborhood V of e such that $u_0 V \subseteq U$. In this case for U^* and V^* as above, we have

$$\overline{u_0 H}^G V^* = \{\overline{u_0 v H}^G : v \in V\} \subseteq \{\overline{uH}^G : u \in U\} = U^*.$$

By continuity of the mapping $\overline{xH}^G \to \overline{a^{-1}H}^G \star \overline{xH}^G \star \overline{aH}^G$ in $\overline{H}^G$, and that $\overline{H}^G \to \overline{a^{-1}H}^G \star \overline{aH}^G = \overline{a^{-1}aH}^G = \overline{H}^G$, we can see for every neighborhood U^* of $\overline{H}^G$, there is a neighborhood V^* of $\overline{H}^G$ such that

$$\overline{a^{-1}H}^G V^* \overline{aH}^G \subseteq U^*.$$

Also for both neighborhoods U^*, V^* of $\overline{H}^G$ as above, let W be a neighborhood of e such that $W \subseteq U \cap V$. Then $W^* = \{\overline{wH}^G : w \in W\}$ is a neghborhood of $\overline{H}^G$ such that

$$W^* \subseteq \{\overline{xH}^G : x \in U \cap V\} \subseteq U^* \cap V^*.$$

Therefore the family of sets $\{\overline{aH}^G U^*\}$, where U^* runs through $\mathscr{U}^*$ and $\overline{aH}^G$ runs through G/H, is an open basis for the quotient topology on G/H. Finally, for each open subset U of G, $\varphi(U) = \{\overline{uH}^G : u \in U\}$ is open in G/H. In fact $\varphi^{-1}(\varphi(U)) = \overline{UH}^G$ and this is clearly open in G. The continuity of φ is virtually obvious. In fact the quotient topology on G/H is the strongest topology on G/H under which the mapping φ is continuous. □

8. The Maximal MI-subgroups

Let G be a topological MI-group. A maximal MI-subgroup H of G is a proper MI-subgroup, such that no proper MI-subgroup K contains H strictly. In other words H is a maximal element of the partially ordered set of proper MI-subgroups of G. In this section we describe some topological properties of the maximal MI-subgroups.

Lemma 8.1. *Let G be a topological MI-group and let M be a maximal MI-subgroup of G. Then M is closed or is densed in G.*

Proof. Let M be a maximal MI-subgroup of G. By theorem 3.3 of [1], $\overline{M}^G$ is also a MI-subgroup of G. Since M is maximal, regarding the relationship $M \subseteq \overline{M}^G \subseteq G$, we have $\overline{M}^G = M$ or $\overline{M}^G = G$. Hence M is closed or is densed in G. □

Theorem 8.2. *Let G be a topological MI-group and let M be a maximal MI-subgroup of G. Then $MP_G = M$ or $MP_G = G$.*

Proof. Let M be a maximal MI-subgroup of G. It is easy to see MP_G is also a MI-subgroup of G. Therefore according to the relationship $M \subseteq MP_G \subseteq G$ and maximality of M, we have $MP_G = M$ or $MP_G = G$, i.e M is open or $MP_G = G$. □

Theorem 8.3. *Let G be a connected topological MI-group. Then for every closed maximal MI-subgroup M of G, $MP_G = G$.*

Proof. By Theorem 8.2, if $MP_G = M$ then M is open and so by assumption, $G = M \cup M^c$ is a separation of G. That's a contradiction. Therefore $MP_G = G$. □

Theorem 8.4. *Let G be a connected topological MI-group. Then every open maximal MI-subgroup M of G is densed in G.*

Proof. By Lemma 8.1, M is closed or is densed in G. if M is closed, then similar to proof the previous Theorem, we will have a separation from G. Hence M is densed in G. □

Theorem 8.5. *Let G be a topological MI-group and P_G be a maximal MI-subgroup of G. Then there is no full maximal MI-subgroups other than P_G.*

Proof. If M be a full maximal MI-subgroup of G then $P_G \subseteq M \subseteq G$. Therefore $M = P_G$ or $M = G$. □

8.1. **The Maximal subgroups of MI-groups.** Let G be a topological MI-group. We consider G as a topological semigroup with identity e satisfying the left and right cancellation laws. In semigroup theory, a maximal subgroup of a semigroup G is a subgroup (a subsemigroup which forms a group under the semigroup operation) of G which is not properly contained in another subgroup of G. There is a one-to-one correspondence between idempotent elements of a semigroup and maximal subgroups of the semigroup: each idempotent element is the identity element of an unique maximal subgroup. But in every MI-group G, if a is an idempotent of G, then $a^2 = a$ and so by the left cancellation law we have $a = e$. Therefore e is the unique idempotent element of G and so there is a unique maximal subgroup of G. Obviously e is the identity element of it.

9. Ideals in MI-group

In every MI-group G as a semigroup with identity e satisfying the left and right cancellation laws, we can define the ideal concept:

Definition 9.1. A nonempty subset M of a MI-group G is called a left ideal (right ideal) if $GM \subseteq M$ ($MG \subseteq M$). If M is a left ideal and a right ideal, M is called a two-sided ideal. A MI-group with no ideals different from G itself is called simple.

Theorem 9.2. *Every simple MI-group is a group.*

Proof. For every $x \neq e$ in G, GxG is clearly an ideal of G. Hence $GxG = G$ and so there are $a, b \in G$ such that $e = axb$. It's easy to see that bax and xba are idempotents. For example $(xba)(xba) = xb(axb)a = xbea = xba$. Since e is the unique idempotent element of G, we have $xba = bax = e$. Therefore, ba is an inverse for x, and G is a group. □

Proposition 9.3. *Let G be a topological MI-group. Then every ideal of G is open.*

Proof. Let I be an ideal of G. Then $IG \subseteq I$ and $GI \subseteq I$. Therefore for each $x \in I$ we have $xP_G \subseteq IG \subseteq I$ and so $I = \{x : xP_G \subseteq I\}$. Hence I is open subset of G. □

Proposition 9.4. *Let G be a connected topological MI-group . Then there is no proper closed ideal of G*

Proof. Let I be a closed ideal of topological MI-group G. By proposition 9.3, I is also open. Therefore since G is a connected topological space, we have to have $I = G$, otherwise $G = I \cup I'$ will be a separation of G. □

10. Automatic continuity in topological MI-groups

In this section, finally, some of the properties of automatic continuity of the homomorphisms of MI-groups are expressed and proved.

Lemma 10.1. *Let $f : G \to G'$ be a homomorphism of MI-groups. Then for every $H \subseteq G$, we have $f(\overline{H}^G) \subseteq \overline{f(H)}^{G'}$. If f is an isomorphism of MI-groups, then $f(\overline{H}^G) = \overline{f(H)}^{G'}$.*

Proof. Let $y \in f(\overline{H}^G)$. Hence there is $x \in \overline{H}^G$ such that $y = f(x)$. Also there exists $s \in P_G$ so that $xs \in H$. Since $f(s) \in P_{G'}$, by relation $f(x)f(s) = f(xs) \in f(H)$, we have $y = f(x) \in \overline{f(H)}^{G'}$. Now suppose that f is one to one and onto. Then for $y \in \overline{f(H)}^{G'}$, there is $t \in P_{G'}$ such that $yt \in f(H)$. Hence there is $x \in H$, $x' \in G$ and $s \in P_G$ such that $yt = f(x)$, $y = f(x')$ and $t = f(s)$. Therefore we will have

$$f(x's) = f(x')f(s) = yt = f(x).$$

Since f is one to one , we conclude that $x's = x$ and so $x' \in \overline{H}^G$. Thus $y = f(x') \in f(\overline{H}^G)$. □

Theorem 10.2. *Let G and G' be two topological MI-groups under the internal topology. Then every homomorphism $f : G \to G'$ is continuous .*

Proof. Suppose that $U \subseteq G'$ is open. Then for each $x \in f^{-1}(U)$, we have $f(x) \in U$. Since U is open subset of G', by definition of open sets in internal topology on G', we get $f(x)P_{G'} \subseteq U$. Hence for every $s \in P_G$ we have $f(xs) = f(x)f(s) \in f(x)P_{G'} \subseteq U$ (We notice that $f(P_G) \subseteq P_{G'}$). Therefore for each $s \in P_G$, we get $xs \in f^{-1}(U)$, i.e. $xP_G \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is an open subset of G. Therefore f is a continuous fuction. □

In general, we will also have:

Theorem 10.3. *Let G' be a topological MI-group under the internal topology and also G be a topological MI-group under the desired topology as satisfying (1) and P_G is open . Then every homomorphism $f : G \to G'$ is continuous .*

Proof. Since each open set U of G' under the internal topology is an union of simple open sets $yP_{G'}$, enough to consider open subsets as $yP_{G'}$. Accordingly, let $x \in f^{-1}(yP_{G'})$. Hence $f(x) \in yP_{G'}$ and so there is $t \in P_{G'}$ such that $f(x) = yt$. Now for each $s \in P_G$ we have

$$f(xs) = f(x)f(s) = ytf(s) \in yP_{G'}.$$

Therefore for every $s \in P_G$ we have $xs \in f^{-1}(yP_{G'})$ or $xP_G \subseteq f^{-1}(yP_{G'})$. By assumption since xP_G is an open subset of G contains x, $f^{-1}(yP_{G'})$ is open. Therefore f is a continuous mapping. □

Theorem 10.4. *Let G and G' be two topological MI-groups under the internal topology. Then every isomorphism $f : G \to G'$ is a closed mapping.*

Proof. Let $f : G \to G'$ be an isomorphism of topological MI-groups G and G' and let H be a closed subset of G, i.e. $\overline{H}^G = H$. Hence by lemma 6 we have $\overline{f(H)}^{G'} = f(\overline{H}^G) = f(H)$. Therefore $f(H)$ is closed in G'. □

According to Theorems 10.2 and 10.4 we have

Theorem 10.5. *Every isomorphism $f : G \to G'$ of MI-groups G and G' is a homeomorphism between them under the internal topology.*

11. Conclusion

In this paper, we defined the concept of a topological MI-group, and then, by defining a particular topology on an arbitrary MI-group, which is called "the internal topology", we determined its various properties. Furthermore, compact subsets, connected components, and open, closed and discrete MI-subgroups of a topological MI-group under the internal topology are identified. In all these results, the role of the MI-subgroup P_G is essential. In fact, this particular MI-subgroup is open in this topology and so each open set is uniquely determined in terms of the right (or left) cosets of P_G. Accordingly, in many of the results and characteristics obtained , the left cosets of P_G have a fundamental role. Indeed, these are the simplest and smallest open sets of this topology.

The authors of this article will review the desired topological MI-groups in their future work. Particularly, the relationship between the properties of topological MI-groups under an arbitrary topology with the same topological MI-groups under the internal topology will be examined. In particular, under an arbitrary topology, if P_G is open, its relation with the internal topology will be specified.

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