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COMMON EXTREMAL GRAPHS FOR THREE INEQUALITIES INVOLVING DOMINATION PARAMETERS

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ABSTRACT. Let $\delta(G)$, $\Delta(G)$ and $\gamma(G)$ be the minimum degree, maximum degree and domination number of a graph $G = (V(G), E(G))$, respectively. A partition of $V(G)$, all of whose classes are dominating sets in G , is called a domatic partition of G . The maximum number of classes of a domatic partition of G is called the domatic number of G , denoted $d(G)$. It is well known that $d(G) \leq \delta(G) + 1$, $d(G)\gamma(G) \leq |V(G)|$ [6], and $|V(G)| \leq (\Delta(G) + 1)\gamma(G)$ [3]. In this paper, we investigate the graphs G for which all the above inequalities become simultaneously equalities.

1. Introduction

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We refer the reader to the book [9] for graph theory notation and terminology not described here. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For a subset $S \subseteq V(G)$ the *subgraph induced by S* is the graph $\langle S \rangle$ with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. The *complement* \overline{G} of G is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G . The square G^2 of a graph G is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most 2. We write K_n for the *complete graph* of order n and C_r for the *cycle* of length r . For any vertex x of a graph G , $N_G(x)$ denotes the set of all neighbors of x in G , $N_G[x] = N_G(x) \cup \{x\}$ and the *degree* of x is $deg_G(x) = |N_G(x)|$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set of vertices D in a graph G is a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex in D . The *domination number* of a graph G , denoted by $\gamma(G)$, is the minimum cardinality

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of a dominating set of G . A dominating set of G is a γ -set, if its cardinality is $\gamma(G)$. A graph G is said to be *excellent* if every vertex belongs to some γ -set. A dominating set D of a graph G is called an *efficient dominating set* (an ED-set) if the distance between any two vertices in D is at least three. Not all graphs have ED-sets. If G has an ED-set, then any ED-set is a γ -set of G [1]. A partition of a nonempty set X is a family of nonempty subsets of X such that every element x in X is in exactly one of these subsets. A *domatic partition* of a graph G is a partition of $V(G)$ into dominating sets. Since $V(G)$ is a dominating set of G , each graph has a domatic partition. The *domatic number* $d(G)$ of G is the maximum number of elements in a domatic partition of G . The concept of domatic number of a graph was introduced by Cockayne and Hedetniemi [6]. A domatic partition of order $d(G)$ is a *d-partition*. The problem of obtaining such a partition is known to be *NP*-complete even for circular arc graphs [2] but can be solved in linear time for interval graphs [14]. A graph G is called *uniquely domatic*, if G has exactly one *d-partition*. A graph G is called *domatically critical* if after deleting an arbitrary edge from G , a graph with a smaller domatic number than that of G is obtained [4]. The domatic partition problem arises in various situations of locating facilities in a network. Assume that a node in a network can access only resources located at neighboring nodes (or at itself). Then if there is an essential type of resource that must be accessible from every node (a hospital, a printer, a file, etc.), copies of the resource need to be distributed over a dominating set of the network. If there are several essential types of resources, each one of them occupies a dominating set. If each node has bounded capacity, there is a limit to the number of resources that can be supported. In particular, if each node can only serve a single resource, the maximum number of resources supportable equals the domatic number of the graph [8].

A set of vertices $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. An *independent dominating set* in G is a set of vertices of G which is both independent and dominating. Partitions into independent dominating sets of $V(G)$ were first considered as particular domatic partitions under the term of *indominable partitions* [5] or *idomatic partitions* [18]. Now the term idomatic is more usual (cf. for instance [15]). Not each graph has an idomatic partition. When an idomatic partition exists on a graph G , then G is called *idomatic* and the *idomatic number* $id(G)$ equals the maximum number of elements in an idomatic partition.

To continue we need the following results.

Theorem A. *For any n -vertex graph G ,*

- (i) [6] $d(G) \leq \delta(G) + 1$ and $d(G)\gamma(G) \leq n$, and
- (ii) [3] $n \leq (\Delta(G) + 1)\gamma(G)$.

In this paper, we mainly turn our attention to the graphs G for which all the inequalities in Theorem A become simultaneously equalities.

2. Results

A graph G without isolated vertices is said to be *δ -edge critical* if $\delta(G) > \delta(G - e)$ for each $e \in E(G)$ [10]. Clearly each k -regular graph, $k \geq 1$, is δ -edge critical. A graph G is said to be *domatically full* if $d(G) = \delta(G) + 1$. All elements of the following classes of graphs are domatically full: (a) trees with

at least 2 vertices [6], (b) outerplanar graphs [6], (c) cycles on $3k$ vertices [6], and (c) strongly chordal graphs [7].

Theorem 2.1. *Let G be a δ -edge critical and domatically full graph. Then the following holds:*

- (i) $d(G) = id(G)$ and each domatic partition of order $d(G)$ is idomatic.
- (ii) If D_1, \dots, D_k is a domatic partition of G with $k = \delta(G) + 1$, then (a) each connected component of the graph $\langle D_i \cup D_j \rangle$ is a star, $i, j = 1, \dots, k$ and $i \neq j$, and (b) if $x \in D_i \cup D_j$ is of minimum degree in G , then x is a leaf of $\langle D_i \cup D_j \rangle$.
- (iii) If G is regular and the partition D_1, \dots, D_k is as in (ii), then $\langle D_i \cup D_j \rangle$ is 1-regular for all $i, j = 1, 2, \dots, k, i \neq j$.
- (iv) G is domatically critical.

Proof. Since G is δ -critical, the set M_δ consisting of all vertices of G having degree more than $\delta(G)$ is independent or empty. Consider any domatic partition D_1, \dots, D_k of G with $k = d(G)$. Since G is domatically full, $k = \delta(G) + 1$. Let $x \in D_i$ and $deg(x) = \delta(G)$. Since each $D_j, j \neq i$, dominates x , each D_j contains exactly one vertex of $N(x)$. Hence x is a leaf of $\langle D_i \cup D_j \rangle$ and D_i is independent. But then D_1, \dots, D_k is an idomatic partition of G and if a vertex $y \in V(\langle D_i \cup D_j \rangle)$ is in M_δ , then all its neighbors are leaves. Thus, (i) and (ii) are satisfied. Clearly, (iii) is an immediate consequence of (ii).

(iv) Since G is δ -edge critical, $\delta(G - e) = \delta(G) - 1$ for each edge e in G . Now by Theorem A, $d(G - e) \leq \delta(G - e) + 1 = \delta(G) \leq d(G) - 1$. Thus, G is domatically critical. □

An *efficient domination partition* (or an *ED-partition*) of a graph G is a partition of $V(G)$ into ED-sets. We say that a graph G is an *efficient domination partitionable graph* (or an *EDP-graph*) if G has an ED-partition. First results on the graphs whose vertex set has a partition in ED-sets are obtained by Mollard in [13]. Clearly, any ED-partition of an EDP-graph is both a domatic partition and an idomatic partition of order $d(G) = id(G)$. A graph G is a *uniquely efficient domination partitioned graph* (or a *UEDP-graph*) if it has only one ED-partition.

The next theorem shows that each regular domatically full graph is an EDP-graph, and vice versa.

Theorem 2.2. *Let G be a graph of order n . Then the following assertions are equivalent.*

- (i) G is an EDP-graph.
- (ii) G is regular and domatically full.
- (iii) $n = \gamma(G)(\Delta(G) + 1)$ and $n = d(G)\gamma(G)$.

Proof. (i) \Rightarrow (ii): Any 2 vertices in the same closed neighborhood of a vertex of G belong to different ED-sets. Hence $d(G) \geq \Delta(G) + 1$. This and Theorem A(i) leads to $\delta(G) = \Delta(G)$ and $d(G) = \delta(G) + 1$. Thus G is regular and domatically full.

(ii) \Rightarrow (iii): By Theorem A, we know that $n \leq \gamma(G)(\Delta(G) + 1)$ and $d(G)\gamma(G) \leq n$. Since $\delta(G) = \Delta(G)$ and $d(G) = \delta(G) + 1$, (iii) is clearly valid.

(iii) \Rightarrow (ii): By (iii), we immediately have $d(G) = \Delta(G) + 1$. But $d(G) \leq \delta(G) + 1$ (Theorem A). Hence (ii) holds.

(ii) \Rightarrow (i): Let D_1, \dots, D_k be a d -domatic partition of G . Hence $k = d(G) = \delta(G) + 1$. By Theorem 2.1, this partition is idomatic and $\langle D_i \cup D_j \rangle$ is 1-regular for all $i, j = 1, 2, \dots, k, i \neq j$. Therefore D_1, \dots, D_k is an ED-partition of G . \square

Corollary 2.3. *Let G be a s -regular graph. If $s \in \{0, 1\}$ then G is an EDP-graph. If $s = 2$ then G is an EDP-graph if and only if the order of each connected component of G is divisible by three.*

Observation 2.4. (Folklore) *Let G be a graph of order n .*

- (i) *If G has an ED-set D whose all vertices have a maximum degree then $n = (\Delta(G) + 1)\gamma(G)$.*
- (ii) *Let $n = (\Delta(G) + 1)\gamma(G)$. Then all γ -sets of G are efficient dominating, and each vertex belonging to some γ -set of G has maximum degree. If G is excellent then G is regular.*

Proof. (i) Obvious.

(ii) Let $D = \{x_1, \dots, x_r\}$ be a γ -set of G . Then

$$(2.1) \quad n = |\cup_{i=1}^r N[x_i]| \leq \sum_{i=1}^r (deg(x_i) + 1) \leq r(\Delta(G) + 1) = \gamma(G)(\Delta(G) + 1).$$

Suppose that $n = \gamma(G)(\Delta(G) + 1)$. Then the inequalities in (2.1) must be equalities. Therefore $N[x_k] \cap N[x_l]$ is empty and $deg(x_i) = \Delta(G)$ for all $k, l, i \in \{1, 2, \dots, r\}$ and $k \neq l$. Thus, D is an ED-set and since D was chosen arbitrarily, each γ -set is an ED-set. The rest is obvious. \square

Proposition 2.5. *Let G be an EDP-graph.*

- (i) [16] *Then G is domatically critical.*
- (ii) [16] *Any domatic partition D_1, \dots, D_k of G , where $k = \delta(G) + 1$, is an ED-partition of G and the graph $\langle D_i \cup D_j \rangle$ is 1-regular, $i, j = 1, \dots, k$ and $i \neq j$.*
- (iii) *Each γ -set of G is efficient dominating.*
- (iv) *If G is an UEDP-graph, then G is uniquely domatic.*

Proof. (i) By Theorem 2.2, G is regular and domatically full. Now Theorem 2.1(iv) implies G is domatically critical.

- (ii) Immediately by Theorem 2.1(iii).
- (iii) Theorem 2.2 and Observation 2.4(ii) together leads to the required.
- (iv) Immediately by (iii). \square

Proposition 2.6. *Let G be an n -order r -regular graph and $n = \gamma(G)(r + 1) = \gamma(\overline{G})(\Delta(\overline{G}) + 1)$. Then $G \in \{K_n, \overline{K}_n\}$.*

Proof. Clearly \overline{G} is a $(n - r - 1)$ -regular graph. If $\gamma(G) = 1$ then $G = K_n$. If $\gamma(G) = n$ then $G = \overline{K}_n$. So, let $n > \gamma(G) \geq 2$. By Observation 2.4, any γ -set of G is efficient dominating. Hence any two vertices in a γ -set of G are at distance at least 3 and must form a γ -set of \overline{G} . Then $n = \gamma(G)(r + 1) = 2(n - r)$, which implies $n = 2r$ and $\gamma(G) = 2 - 2/(r + 1)$, a contradiction. \square

Corollary 2.7. *The graphs G and \overline{G} are both EDP-graphs if and only if one of them is complete.*

3. Examples

In this section we present some examples of EDP-graphs. A *crown graph* $H_{n,n}$, is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Example 3.1. Let $G = H_{n,n}$, $n \geq 3$, $V(G) = \{v_i, u_i \mid i = 1, 2, \dots, n\}$ and $E(G) = \{v_i u_j \mid i \neq j\}$. Then all γ -sets of G are $\{u_i, v_i\}$, $i = 1, 2, \dots, n$. Obviously, they form the unique ED-partition of G . Thus, G is an UEDP-graph.

Denote by $\mathbb{Z}_n = \{0, 1, \dots, n - 1\}$ the additive group of order n . Let S be a subset of \mathbb{Z}_n such that $0 \notin S$ and $x \in S$ implies $-x \in S$. The *circulant graph* with distance set S is the graph $C(n; S)$ with vertex set \mathbb{Z}_n and vertex x adjacent to vertex y if and only if $x - y \in S$. It is clear from the definition that $C(n; S)$ is vertex-transitive and regular of degree $|S|$.

Example 3.2. Let $G = C(n = (2k + 1)t; \{1, \dots, k\} \cup \{n - 1, \dots, n - k\})$, where $k, t \geq 1$. Denote by D_r the set all elements of which are $r, r + (2k + 1), \dots, r + (2k + 1)(t - 1)$, where addition is taken mod $(2k + 1)t$ and $r \in \{0, \dots, (2k + 1)t - 1\}$. Clearly all D_r 's are ED-sets, and D_0, D_1, \dots, D_{2k} is the unique ED-partition of G . Thus, G is an UEDP-graph.

Example 3.3. Let $G = C(n; \{\pm 1, \pm s\})$ where $2 \leq s \leq n - 2$ and $s \neq n/2$. Then G has an ED-set if and only if $5|n$ and $s \equiv \pm 2 \pmod{5}$; in addition, all ED-sets in G have the form $D_i = \{v \in V(G) \mid v \equiv i \pmod{5}\}$ [11]. Thus (a) G is an EDP-graph if and only if $5|n$ and $s \equiv \pm 2 \pmod{5}$, and (b) if G is an EDP-graph then G is an UEDP-graph.

Let $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$. The *generalized Petersen graph* $P(n, k)$ is the graph on the vertex-set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ with adjacencies $x_i x_{i+1}$, $x_i y_i$, and $y_i y_{i+k}$ for all i . The graph $P(n, 1)$ is equivalent to the n -prism.

Example 3.4. A graph $P(n, k)$ is an EDP-graph if and only if $n \equiv 0 \pmod{4}$ and k is odd. If $P(n, k)$ is an EDP-graph, then $P(n, k)$ is an UEDP-graph. In particular, an n -prism is an UEDP-graph if and only if $n \equiv 0 \pmod{4}$.

Proof. A generalized Petersen graph $P(n, k)$ has an ED-set if and only if $n \equiv 0 \pmod{4}$ and k is odd (Theorem 1 [12]). Moreover, by the proof of this theorem it follows that each vertex of $P(n, k)$ belongs to exactly one ED-set. Let $n = 4r$ and k odd. We construct the only ED-partition D_0, D_1, D_2, D_3 of $P(n, k)$ as follows: $D_s = \{u_{4i+1+s} \mid 0 \leq i \leq r - 1\} \cup \{v_{4i+3+s} \mid 0 \leq i \leq r - 1\}$, $s = 0, 1, 2, 3$. Thus $P(n, k)$ is an UEDP-graph. The rest is obvious. \square

For two graphs G_1 and G_2 , the *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$ if and only if $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$ or $x_2 y_2 \in E(G_2)$ and $x_1 = y_1$. The *Cartesian power* $G^{\square n}$ of a graph G is the graph recursively defined by $G^{\square 1} = G$, and $G^{\square n} = G^{\square(n-1)} \square G$ for $d > 1$. The *hypercube of dimension n* is the graph $Q_n = K_2^{\square n}$.

Example 3.5. The following result (reformulated in our present terminology) is from [13].

- (i) Let G and H be two n -regular EDP-graphs and let H be bipartite. Then $G \square H \square P_2$ is an EDP-graph.
- (ii) Let G be a bipartite EDP-graph. Then $G^{\square 2^k} \square Q_{2^{k-1}}$ is an EDP-graph, $k \geq 1$.

The wreath product of graphs G and H is the graph, $G \text{ wr } H$ with vertex set $V(G \text{ wr } H) = V(G) \times V(H)$ and edge set $E(G \text{ wr } H) = \{(x, y)(v, w) \mid xv \in E(G), \text{ or } x = v \text{ and } yw \in E(H)\}$. Informally, $G \text{ wr } H$ is the graph obtained by replacing each vertex of G by a copy of H and putting all possible edges between copies of H that replaced adjacent vertices of G .

Example 3.6. If G is an EDP-graph, then $G \text{ wr } K_r$ is also an EDP-graph.

Proof. Let $(D_i = \{x_i^1, x_i^2, \dots, x_i^s\})_{i=1}^k$ be an ED-partition of G , and let $V(K_r) = \{y_1, \dots, y_r\}$. Then $(U_i^j = \{(x_i^1, y_j), (x_i^2, y_j), \dots, (x_i^s, y_j)\})_{i=1}^k, j=1}^r$ is an ED-partition of $G \text{ wr } K_r$. □

4. EDP-GRAPHS OF ORDER AT MOST 10

We say that the partitions $(D_i)_{i=1}^r$ and $(U_j)_{j=1}^s$ of a set X are *orthogonal* whenever $|D_i \cap U_j| = 1$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$.

The next observation is obvious but useful in the sequel.

Observation 4.1. Let $\pi = (D_i = \{x_i^1, x_i^2, \dots, x_i^s\})_{i=1}^k$ be an ED-partition of an EDP-graph G .

- (A₁) Then $\sigma_i = (N_G[x_i^j])_{j=1}^s$ is a partition of $V(G)$ which is orthogonal to π , $i = 1, 2, \dots, k$. If $k \geq 2$ then $G - D_i$ is an EDP-graph, $\pi - D_i$ is an ED-partition of $V(G - D_i)$, and $(N_G(x_i^j))_{j=1}^s$ is a partition of $V(G - D_i)$ which is orthogonal to $\pi - D_i$.
- (A₂) Let $\tau = (U_j)_{j=1}^s$ be a partition of $V(G)$ which is orthogonal to π . Define the graph G' as obtained from G by adding s new vertices u_1, u_2, \dots, u_s and $s(\Delta(G) + 1)$ new edges, so that G is an induced subgraph of G' and $N_{G'}(u_j) = U_j$. Then G' is an EDP-graph, $\pi' = \pi \cup \{u_1, u_2, \dots, u_s\}$ is an ED-partition of G' , and $\tau' = (U_j \cup \{u_j\})_{j=1}^s$ is a partition of G' which is orthogonal to π' .
- (A₃) $G_r = \langle \cup_{i=1}^r D_i \rangle$ is an EDP-graph for all $r = 1, \dots, k$.

We next give some applications of Observation 4.1. Denote by EDP_n the set of all n -vertex mutually non-isomorphic EDP-graphs.

Theorem 4.2. Let n be a positive integer.

- (i) $\overline{K}_n, K_n \in \text{EDP}_n$ and whenever n is prime, $\text{EDP}_n = \{\overline{K}_n, K_n\}$.
- (ii) $\text{EDP}_4 = \{\overline{K}_4, 2K_2, K_4\}$.
- (iii) $\text{EDP}_6 = \{\overline{K}_6, 3K_2, 2K_3, C_6, K_6\}$.
- (iv) EDP_8 consists of $\overline{K}_8, 4K_2, 2K_4, H_{4,4} = Q_3, K_8$ and the 3-regular connected graph G_0 obtained from 2 disjoint copies of $K_4 - e$ by adding 2 edges.
- (v) $\text{EDP}_9 = \{\overline{K}_9, 3K_3, K_3 \cup C_6, C_9, K_9\}$.
- (vi) EDP_{10} consists of $\overline{K}_{10}, 5K_2, 2K_5, H_{5,5} = C(10; \{\pm 1, \pm 3\}), K_{10}$ and the graphs G_1, \dots, G_5 depicted in Fig. 1.

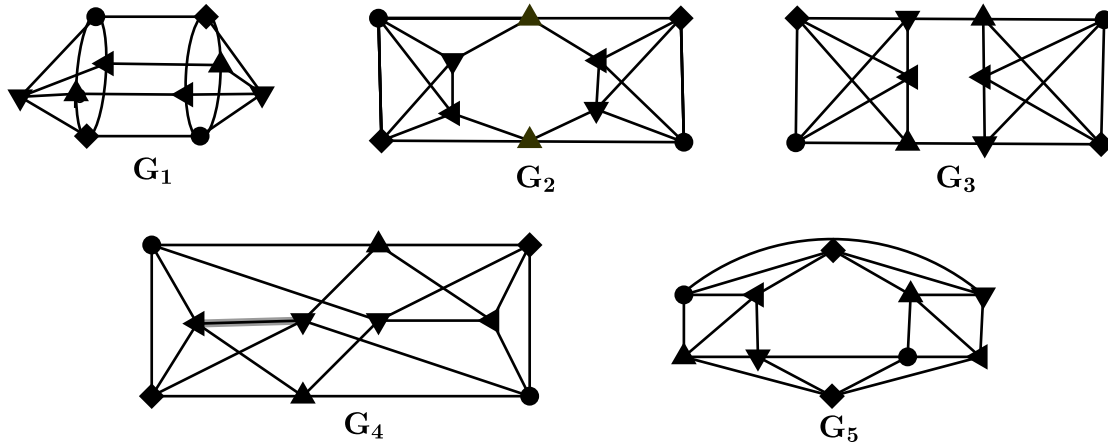


FIGURE 1. Graphs G_1, \dots, G_5 .

Proof. First note that all graphs mentioned in (i)–(vi) are clearly EDP-graphs. Consider any $G \in \text{EDP}_n$. We know by Theorem 2.2 that G is regular and $n = \gamma(G)d(G) = \gamma(G)(\Delta(G) + 1)$. Hence if n is prime, then either $\Delta(G) = 0$ and $G = \overline{K}_n$, or $\Delta(G) = n - 1$ and $G = K_n$. Thus, (i) holds.

In what follows, let $n \in \{4, 6, 8, 9, 10\}$. If $\Delta(G) \in \{0, 1, 2\}$ then $G \in \{\overline{K}_4, 2K_2, \overline{K}_6, 3K_2, 2K_3, C_6, \overline{K}_8, 4K_2, \overline{K}_9, 3K_3, K_3 \cup C_6, C_9\}$, because of Corollary 2.3. If $\Delta(G) = n - 1$ then clearly $G = K_n$. So, let $3 \leq \Delta(G) \leq n - 2$. Since $\Delta(G) + 1$ is a divisor of n , only the following 2 cases are possible: (a) $n = 8$ and $\Delta(G) = 3$, and (b) $n = 10$ and $\Delta(G) = 4$.

Case 1: $n = 8$ and $\Delta(G) = 3$. Hence $\gamma(G) = 2$ and $d(G) = 4$. Let D_1, D_2, D_3, D_4 be any ED-partition of G . By Observation 4.1, $G - D_4$ is a 2-regular 6-vertex EDP-graph; hence by (iii), $G - D_4 \in \{2K_3, C_6\}$. It is easy to see that $G \in \{2K_4, G_0\}$ when $G - D_4 = 2K_3$, and $G \in \{G_0, Q_3\}$ when $G - D_4 = C_6$. Thus, (iv) holds.

Case 2: $n = 10$ and $\Delta(G) = 4$. Hence $\gamma(G) = 2$ and $d(G) = 5$. Consider any ED-partition D_1, \dots, D_5 of G . Note that $G - D_5$ is a 3-regular 8-vertex EDP-graph because of Observation 4.1. Now by (iv), $G - D_5 \in \{2K_4, Q_3, G_0\}$. It is not hard to see that (a) $G \in \{2K_5, G_2, G_3\}$ when $G - D_5 = 2K_4$, (b) $G \in \{G_1, \dots, G_5\}$ when $G - D_5 = G_0$, and (c) $G \in \{G_1, G_4, H_{5,5}\}$ when $G - D_5 = Q_3$. Thus, (vi) is valid. \square

Proposition 4.3. *Let G be a s -regular EDP-graph of order n , where $n - 1 > s \geq 3$.*

- (i) *Then $s \notin \{n - 3, n - 2\}$.*
- (ii) *$s = n - 4$ if and only if $G = C_6$.*
- (iii) *$s = n - 5$ if and only if $G \in \{\overline{K}_5, 3K_2, 2K_4, Q_3, G_0\}$.*
- (iv) *$s = n - 6$ if and only if $G \in \{\overline{K}_6, 2K_5, H_{5,5}, G_1, \dots, G_5\}$.*
- (v) *If $n - s - 1$ is a prime, then $n = 2s + 2$ and $\gamma(G) = 2$.*

Proof. Obviously $n \geq 5$. By Theorem 2.2, $n = \gamma(G)(s + 1) = d(G)\gamma(G)$ and $d(G) = s + 1$. Hence $n/(s + 1)$ is an integer. But then (i) holds. Note now that all graphs mentioned in (ii)–(vi) are clearly EDP-graphs.

(ii) Since $n/(n-3)$ is an integer and $n \geq 5$, we have $n = 6$ and $s = 2$. Hence $G = C_6$.

(iii) $n/(n-4)$ is an integer implies that either ($n = 5$, $s = 0$ and then $G = \overline{K_5}$), or ($n = 6$, $s = 1$ and $G = 3K_2$), or ($s = 3$ and $G \in \text{EDP}_8$). The result follows by Theorem 4.2(iv).

(iv) As $n/(n-5)$ is an integer, either $G = \overline{K_6}$ or ($s = 4$ and $G \in \text{EDP}_{10}$). By Theorem 4.2(vi) we immediately obtain the required.

(v) Since $p = n - s - 1$ is a prime and $\gamma(G) = n/(s+1) = 1 + p/(s+1)$, it follows that either G is edgeless or ($p = s + 1$, $\gamma(G) = 2$ and $n = 2p$). \square

5. UNIQUELY COLORABLE GRAPHS

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G . Any such minimum partition is called a χ -*partition* of $V(G)$. A graph G is called *uniquely $\chi(G)$ -colorable* if G has exactly one χ -partition. Each member of the only χ -partition of a uniquely $\chi(G)$ -colorable graph G is an independent dominating set of G ; hence G is idomatic [5]. We need the following result, due to Zelinka [17].

Theorem B. [17] *Let G be a regular and domatically full graph. Then $d(G) = \chi(G^2)$ and each χ -partition of G^2 is a domatic partition of G with $d(G)$ members and vice versa. Furthermore, G is uniquely domatic if and only if G^2 is uniquely $\chi(G^2)$ -colorable.*

Theorem 5.1. *Let G be a graph for which one of the following holds.*

- (i) $G = H_{n,n}$, $n \geq 3$.
- (ii) $G = C(n = (2k+1)t; \{1, \dots, k\} \cup \{n-1, \dots, n-k\})$, where $1 \leq k \leq (n-1)/2$ and $t \geq 1$.
- (iii) Let $G = C(n; \{\pm 1, \pm s\})$, where $5|n$, $2 \leq s \leq n-2$, $s \equiv \pm 2 \pmod{5}$ and $s \neq n/2$.
- (iv) $G = P(n, k)$, where $n \equiv 0 \pmod{4}$ and k is odd.

Then G^2 is uniquely $\chi(G^2)$ -colorable and $d(G) = \chi(G^2)$.

Proof. Examples 3.1–3.4 show that all graphs in (i)–(iv) are UEDP-graphs. By Proposition 2.5(iv) all these graphs are uniquely domatic. The result now immediately follows by Theorem B. \square

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