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ON ALMOST RECOGNIZABILITY BY SPECTRUM OF SIMPLE CLASSICAL GROUPS

ALEXEY STAROLETOV

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ABSTRACT. The set of element orders of a finite group G is called the *spectrum*. Groups with coinciding spectra are said to be *isospectral*. It is known that if G has a nontrivial normal soluble subgroup then there exist infinitely many pairwise non-isomorphic groups isospectral to G . The situation is quite different if G is a nonabelian simple group. Recently it was proved that if L is a simple classical group of dimension at least 62 and G is a finite group isospectral to L , then up to isomorphism $L \leq G \leq \text{Aut } L$. We show that the assertion remains true if 62 is replaced by 38.

1. Introduction

Given a finite group G , the *spectrum* of G is the set of its element orders and is denoted by $\omega(G)$. Finite groups are said to be isospectral if their spectra coincide. It is known that if a finite group G has a nontrivial normal soluble subgroup then there exist infinitely many pairwise non-isomorphic groups isospectral to G (see [9, 11]). In contrast, there are a lot of finite nonabelian simple groups that are uniquely determined by their spectra in the class of finite groups. Such groups are said to be *recognizable* (by spectrum). Moreover, recently it was proved that every "sufficiently large" nonabelian simple group L is *almost recognizable* (by spectrum), that is there are finitely many pairwise non-isomorphic finite groups isospectral to L . Namely, the following holds (see [6, 12, 16]).

Theorem A. *Let L be one of the following nonabelian simple groups:*

- (1) *a sporadic group other than J_2 ;*
- (2) *an alternating group A_n , where $n \neq 6, 10$;*
- (3) *an exceptional group of Lie type other than ${}^3D_4(2)$;*
- (4) *$L_n(q)$, where $n \geq 45$ or q is even;*
- (5) *$U_n(q)$, where $n \geq 45$, or q is even and $(n, q) \neq (4, 2), (5, 2)$;*

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- (6) $S_{2n}(q)$, $O_{2n+1}(q)$, where $n \geq 29$, or q is even, $n \neq 2, 4$, and $(n, q) \neq (3, 2)$;
- (7) $O_{2n}^+(q)$, where $n \geq 31$, or q is even and $(n, q) \neq (4, 2)$;
- (8) $O_{2n}^-(q)$, where $n \geq 30$ or q is even.

Then every finite group isospectral to L is isomorphic to some group G with $L \leq G \leq \text{Aut } L$. In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to L .

Remark 1.1. In the statements of Theorem 2 and Theorem 3 in [12], the condition for $S_{2n}(q)$ and $O_{2n+1}(q)$ is $n \geq 28$, but in fact these theorems are proved for $n \geq 29$.

Besides the theorem, A. Vasil'ev and M. Grechkoseeva in [6] formulated the following.

Conjecture B. Let L be one of the following groups:

- (1) $L_n(q)$, where $n \geq 5$;
- (2) $U_n(q)$, where $n \geq 5$ and $(n, q) \neq (5, 2)$;
- (3) $S_{2n}(q)$, where $n \geq 3$, $n \neq 4$ and $(n, q) \neq (3, 2)$;
- (4) $O_{2n+1}(q)$, where q is odd, $n \geq 3$, $n \neq 4$ and $(n, q) \neq (3, 3)$;
- (5) $O_{2n}^\varepsilon(q)$, where $n \geq 4$ and $(n, q, \varepsilon) \neq (4, 2, +), (4, 3, +)$.

Then every finite group isospectral to L is isomorphic to some group G with $L \leq G \leq \text{Aut } L$.

In Theorem A, Conjecture B, and throughout the paper, we use single-letter names for simple classical groups, following [3]. Also we use the standard abbreviation $L_n^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, $L_n^+(q) = L_n(q)$, and $L_n^-(q) = U_n(q)$. We continue investigation of the recognition problem for simple classical group and weaken restrictions on classical groups in Theorem A.

Theorem 1.2. Let L be one of the simple groups $L_n^\varepsilon(q)$ with $n \geq 27$, $S_{2n}(q)$, $O_{2n+1}(q)$ with $n \geq 16$, $L = O_{2n}^+(q)$ with $n \geq 19$, and $L = O_{2n}^-(q)$ with $n \geq 18$. Then every finite group isospectral to L is isomorphic to some group G with $L \leq G \leq \text{Aut } L$. In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to L .

In fact, as we will show in the last section, Theorem 1.2 is a straightforward consequence of a series of previous results and the following theorem whose proof is the main goal of this paper.

Theorem 1.3. Let q be a power of an odd prime p , and let L be one of the groups $L_n^\varepsilon(q)$ with $27 \leq n \leq 44$, $S_{2n}(q)$, $O_{2n+1}(q)$ with $17 \leq n \leq 28$, $O_{2n}^+(q)$ with $19 \leq n \leq 30$, and $O_{2n}^-(q)$ with $18 \leq n \leq 29$. If $\omega(G) = \omega(L)$ for a finite group G , then G has a unique nonabelian composition factor S , and S is not isomorphic to a group of Lie type over a field of characteristic distinct from p .

Observe that Theorems 1.2 and 1.3 resemble Theorem 2 and Theorem 3 in [12] but their hypotheses are weaker. No wonder the structure of the present paper is similar to that of [12]. Moreover, we use a series of lemmas from [12] and either give a new proof to enhance their assertions or explain how to change the original proof to obtain required conclusions under weaker assumptions.

2. Preliminaries: arithmetic of Zsigmondy primes

Given a set of nonzero integers n_1, \dots, n_k , we denote by (n_1, \dots, n_k) and $[n_1, \dots, n_k]$ their greatest common divisor and least common multiple, respectively. Given a nonzero integer n , we put $\varphi(n)$ for the Euler totient function of n , $\pi(n)$ for the set of prime divisors of n , and if G is a finite group then, as usual, $\pi(G)$ stands for $\pi(|G|)$. If π is a set of primes, then n_π denotes the π -part of n , that is, the largest divisor k of n with

$\pi(k) \subseteq \pi$; and $n_{\pi'}$ denotes the π' -part of n , that is, the ratio $|n|/n_{\pi}$. If n is a nonzero integer and r is an odd prime with $(r, n) = 1$, then $e(r, n)$ denotes the multiplicative order of n modulo r . Given an odd integer n , we put $e(2, n) = 1$ if $n \equiv 1 \pmod{4}$, and $e(2, n) = 2$ otherwise.

Fix an integer a with $|a| > 1$. A prime r is said to be a *primitive prime divisor* of $a^i - 1$ if $e(r, a) = i$. We write $r_i(a)$ to denote some primitive prime divisor of $a^i - 1$, if such a prime exists, and $R_i(a)$ to denote the set of all such divisors. Zsigmondy [24] proved that primitive prime divisors exist for almost all pairs (a, i) .

Lemma 2.1. (Zsigmondy [24]) *Let a be an integer and $|a| > 1$. For every natural number i the set $R_i(a)$ is nonempty, except for the pairs $(a, i) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}$.*

Let $i \neq 2$ be a positive integer. Then the product of all primitive prime divisors of $a^i - 1$ taken with multiplicities is denoted by $k_i(a)$. Put $k_2(a) = k_1(-a)$. The number $k_i(a)$ is said to be the *greatest primitive divisor* of $a^i - 1$. It follows from the definition that $(k_i(a), k_j(a)) = 1$ if $i \neq j$. It is easy to check that $k_1(a) = |a - 1|$ if $a \not\equiv 3 \pmod{4}$, and $k_1(a) = |a - 1|/2$ if $a \equiv 3 \pmod{4}$, as well as $k_2(a) = |a + 1|$ if $a \not\equiv 1 \pmod{4}$, and $k_2(a) = |a + 1|/2$ if $a \equiv 1 \pmod{4}$. It follows from [10] that for $i > 2$,

$$(2.1) \quad k_i(a) = \frac{|\Phi_i(a)|}{(r, \Phi_{i_{\{r\}'}}(a))},$$

where $\Phi_i(x)$ is the i th cyclotomic polynomial and r is the largest prime dividing i ; moreover, if $i_{\{r\}'}$ does not divide $r - 1$ then $(r, \Phi_{i_{\{r\}'}}(a)) = 1$.

Lemma 2.2. ([12, Lemma 1.3]) *Let a and i be integers with $|a| > 1$ and $i > 0$. If i is odd then $k_i(-a) = k_{2i}(a)$, and if i is a multiple of 4 then $k_i(-a) = k_i(a)$.*

Lemma 2.3. *Let $n < 105$. Then all coefficients of $\Phi_n(x)$ belong to the set $\{-1, 0, 1\}$. In particular, for all $x > 0$ it is true that $x^{\varphi(n)} - x^{\varphi(n)-1} - \dots - x - 1 \leq \Phi_n(x) \leq x^{\varphi(n)} + x^{\varphi(n)-1} + \dots + x + 1$. As a consequence we have that if $x \geq 2$ then $\Phi_n(x) \leq 2x^{\varphi(n)}$ and if $x \geq 3$ then $x^{\varphi(n)}/2 \leq \Phi_n(x)$.*

Proof. The fact that all coefficients of $\Phi_n(x)$ belong to the set $\{-1, 0, 1\}$ if $n < 105$ is well-known and mentioned for example in [4]. Also it is known that $\deg(\Phi_n(x)) = \varphi(n)$, whence $x^{\varphi(n)} - x^{\varphi(n)-1} - \dots - x - 1 \leq \Phi_n(x) \leq x^{\varphi(n)} + x^{\varphi(n)-1} + \dots + x + 1$ for all $x > 0$. If $x \geq 2$ then $x^{\varphi(n)} \geq x^{\varphi(n)-1} + x^{\varphi(n)-2} + \dots + x + 1$, so $\Phi_n(x) \leq 2x^{\varphi(n)}$. If $x \geq 3$ then $x^{\varphi(n)} \geq 3x^{\varphi(n)-1} \geq 2x^{\varphi(n)-1} + x^{\varphi(n)-1} \geq 2x^{\varphi(n)-1} + 2x^{\varphi(n)-2} + x^{\varphi(n)-2} \geq \dots \geq 2x^{\varphi(n)-1} + 2x^{\varphi(n)-2} + \dots + 2x + 2$, so $x^{\varphi(n)}/2 \leq \Phi_n(x)$. □

Lemma 2.4. ([12, Lemma 1.5]) *Let a and i be integers, and $\varepsilon \in \{+, -\}$. If $a \geq 2$, $i \geq 3$, and $(a, i) \notin \{(2, 3), (2, 6)\}$, then $k_i(\varepsilon a) > a^{\varphi(i)/2}$.*

Lemma 2.5. *Let q and u be prime powers, where $q \neq u$ and q is odd. Then the following hold.*

- (i) *If $k_{2i}(q)$ divides $k_{2i}(u)$ and $i \in \{8, 12\}$ then $iq^i < u^i$.*
- (ii) *If $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$ then $3q^{12} < u^{12}$.*
- (iii) *If $k_9(\varepsilon q)$ divides $k_9(\tau u)$, where $\varepsilon, \tau \in \{+, -\}$ then $3q^6 < u^6$.*
- (iv) *If $k_7(\varepsilon q)$ divides $k_7(\tau u)$, where $\varepsilon, \tau \in \{+, -\}$, and $(q - \varepsilon 1, 7) = 1$ then $3q^6 < u^6$.*

Proof. Let $k_{16}(q)$ divides $k_{16}(u)$. First we prove that $k_{16}(q) \neq k_{16}(u)$. Assume the contrary. According to (2.1), we obtain that $(q^8 + 1)/2 = (u^8 + 1)/(u - 1, 2)$. If $(u - 1, 2) = 2$ then $u = q$; a contradiction. So $(u - 1, 2) = 1$, and hence $q^8 = 2u^8 + 1$. Since u is even, we have that $(u, 5) = 1$. Then $u^4 \equiv 1 \pmod{5}$, and

hence $2u^8 + 1 \equiv 3 \pmod{5}$. It is clearly that $q^8 \equiv 0, 1 \pmod{5}$, so we get a contradiction. Thus $k_{16}(u)/k_{16}(q) > 1$. By Fermat's little theorem, we have that $r_{16}(u) \equiv 1 \pmod{16}$, therefore $17k_{16}(q) \leq k_{16}(u)$, and hence $(17/2)(q^8 + 1) \leq u^8 + 1$. So $8q^8 < u^8 - 7 < u^8$.

Let $k_{24}(q)$ divides $k_{24}(u)$. Since for an integer x it is true that $k_{24}(x) = \Phi_{24}(x) = x^8 - x^4 + 1$, the equality $k_{24}(u) = k_{24}(q)$ is equivalent to $u = q$. Note that $r_{24}(u) \equiv 1 \pmod{24}$, so $25k_{24}(q) \leq k_{24}(u)$. It is clear that $k_{24}(u) < u^8$ and $q^8/2 < k_{24}(q)$, whence $12q^8 < u^8$, as required.

Now we prove (ii). Assume that $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$. If x is an integer then $k_9(x)k_{18}(x) = \frac{x^6+x^3+1}{(x-1,3)} \frac{x^6-x^3+1}{(x+1,3)} = \frac{x^{12}+x^6+1}{(x-1,3)(x+1,3)}$ and clearly $(x-1,3)(x+1,3) \in \{1,3\}$. Suppose that $k_9(q)k_{18}(q) = k_9(u)k_{18}(u)$. If $(q-1,3)(q+1,3) = (u-1,3)(u+1,3)$ then $q^{12} + q^6 + 1 = u^{12} + u^6 + 1$, and hence $u = q$; a contradiction. So we may assume that $(q-1,3)(q+1,3) = 1, (u-1,3)(u+1,3) = 3$, the other case is considered similarly. Then $3(q^{12} + q^6 + 1) = u^{12} + u^6 + 1$, and hence $3q^6(q^6 + 1) = (u^6 + 2)(u^6 - 1)$. Since $(u^6 + 2, u^6 - 1) \in \{1,3\}$, we have that either $u^6 + 2$ or $u^6 - 1$ is divisible by q^6 . It is clear that $u^6 + 2 \neq q^6$ and $u^6 - 1 \neq q^6$, therefore $(u^6 + 2)(u^6 - 1) \geq 2q^6(2q^6 - 3)$. So $3(q^{12} + q^6 + 1) \geq 4q^{12} - 6q^6$, and hence $q^{12} \leq 9q^6 + 3$; a contradiction since this inequality is not true if $q \geq 2$. Thus $k_9(q)k_{18}(q)$ is a proper divisor of $k_9(u)k_{18}(u)$. Since $r_9(u) \equiv 1 \pmod{18}$ and $r_{18}(u) \equiv 1 \pmod{18}$, we have that $k_9(u)k_{18}(u) \geq 19k_9(q)k_{18}(q)$, and hence $2u^{12} > u^{12} + u^6 + 1 \geq (19/3)(q^{12} + q^6 + 1) > 6(q^{12} + q^6 + 1)$. Therefore $u^{12} > 3q^{12}$, as claimed.

Now we prove (iii). Assume that $k_9(\varepsilon q)$ divides $k_9(\tau u)$, where $\varepsilon, \tau \in \{+, -\}$. Note that $k_9(\varepsilon q) = (q^6 + \varepsilon q^3 + 1)/(q - \varepsilon 1, 3)$, $k_9(\tau u) = (u^6 + \tau u^3 + 1)/(u - \tau 1, 3)$. Assume at first that $(q - \varepsilon 1, 3) = (u - \tau 1, 3)$. Then $q^6 + \varepsilon q^3 + 1$ divides $u^6 + \tau u^3 + 1$. Note that $q^6 + \varepsilon q^3 + 1 \neq u^6 + \tau u^3 + 1$, since otherwise either $u = q$ or $u^3 \pm q^3 = \pm 1$, which is impossible. Since $r_9(\tau u) \geq 19$, we have that $19(q^6 + \varepsilon q^3 + 1) \leq u^6 + \tau u^3 + 1$. Now $(19/2)q^6 \leq 19(q^6 + \varepsilon q^3 + 1) \leq u^6 + \tau u^3 + 1 \leq 2u^6$, and hence $4q^6 < u^6$, in particular, $3q^6 < u^6$, as required. So we may assume that $(q - \varepsilon 1, 3) \neq (u - \tau 1, 3)$. We have that $\frac{(u - \tau 1, 3)}{(q - \varepsilon 1, 3)}(q^6 + \varepsilon q^3 + 1)$ divides $(u^6 + \tau u^3 + 1)$. First we prove that these numbers are not equal. Assume the contrary and consider the case $(q - \varepsilon 1, 3) = 1, (u - \tau 1, 3) = 3$, since the other case is symmetric to this one. So $3q^3(q^3 + \varepsilon 1) = (u^3 - \tau 1)(u^3 + \tau 2)$. Now $(u^3 - \tau 1, u^3 + \tau 2) = 3$, so only one of them is divisible by q^3 . Since $u^3 - \tau 1 \neq q^3$ and $u^3 + \tau 2 \neq q^3$, we have that either $u^3 - \tau 1 \geq 2q^3$ or $u^3 + \tau 2 \geq 2q^3$. Therefore $(u^3 - \tau 1)(u^3 + \tau 2) \geq 2q^3(2q^3 - 3)$. So $3q^3(q^3 + \varepsilon 1) \geq 4q^6 - 6q^3$, and hence $(6 + \varepsilon 3)q^3 \geq q^6$; a contradiction, since $q \geq 3$. In the symmetric case $(q^3 - \varepsilon 1)(q^3 + \varepsilon 2) = 3u^3(u^3 + \tau 1)$ and we obtain that $(6 + \tau 3)u^3 \geq u^6$. Whence $u = 2$ and $\tau = +$, and hence $(q^3 - \varepsilon 1)(q^3 + \varepsilon 2) = 216$; a contradiction, since $q \geq 3$ and $(q^3 - \varepsilon 1)(q^3 + \varepsilon 2) > 216$. So we obtain that $\frac{(u - \tau 1, 3)}{(q - \varepsilon 1, 3)}(q^6 + \varepsilon q^3 + 1)$ is a proper divisor of $(u^6 + \tau u^3 + 1)$. Since $r_9(\tau u) \geq 19$, we have that $\frac{19(u - \tau 1, 3)}{(q - \varepsilon 1, 3)}(q^6 + \varepsilon q^3 + 1) \leq (u^6 + \tau u^3 + 1)$. So $(19/3)(q^6 + \varepsilon q^3 + 1) \leq (u^6 + \tau u^3 + 1)$. Observe that $18q^6 < 19(q^6 + \varepsilon q^3 + 1)$, since $q \geq 3$. Thus $6q^6 \leq (u^6 + \tau u^3 + 1) \leq 2u^6$, and hence $3q^6 < u^6$, as required.

Now we prove (iv). Assume that $k_7(\varepsilon q)$ divides $k_7(\tau u)$, where $\varepsilon, \tau \in \{+, -\}$, and $(q - \varepsilon 1, 7) = 1$. Then $k_7(\varepsilon q) = q^6 + \varepsilon q^5 + q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$ and $k_7(\tau u) = \frac{u^6 + \tau u^5 + u^4 + \tau u^3 + u^2 + \tau u + 1}{(u - \tau 1, 7)}$. Suppose that $k_7(\varepsilon q) = k_7(\tau u)$. Since $u \neq q$, either $q \geq u + 1$ or $u \geq q + 1$ and hence if $(u - \tau 1, 7) = 1$ then we obtain that either $k_7(\varepsilon q) > k_7(\tau u)$ or $k_7(\tau u) > k_7(\varepsilon q)$ respectively. So $(u - \tau 1, 7) = 7$, and hence $u \geq 8$. Since $q \geq 3$, we have that $k_7(\varepsilon q) \geq q^6 - q^5 + q^4 - q^3 + q^2 - q + 1 > (2/3)q^6$, and since $u \geq 8$, we obtain that $k_7(\tau u) \leq (u^6 + u^5 + u^4 + u^3 + u^2 + u + 1)/7 \leq (3/14)u^6$. Therefore $(28/9)q^6 \leq u^6$, and hence $3q^6 < u^6$, as required. Thus $k_7(\varepsilon q)$ is a proper divisor of $k_7(\tau u)$. By Fermat's little theorem, for any prime r dividing $k_7(\tau u)$, we have that $r - 1$ is divisible by 7. Therefore $r \geq 29$, and hence $29k_7(\varepsilon q) \leq k_7(\tau u)$. By Lemma 2.3, we have that $k_7(\varepsilon q) \geq q^6/2$ and $k_7(\tau u) \leq 2u^6$. So $(29/4)q^6 \leq u^6$, in particular $3q^6 < u^6$. The lemma is proved. \square

Define the following function on positive integers:

$$(2.2) \quad \eta(k) = \begin{cases} k, & \text{if } k \text{ is odd,} \\ k/2, & \text{if } k \text{ is even.} \end{cases}$$

Lemma 2.6. *Let n be a positive integer. Then $\varphi(n) = 6$ is equivalent to $n \in \{7, 9, 14, 18\}$ and $\varphi(n) = 8$ is equivalent to $n \in \{15, 16, 20, 24, 30\}$.*

Proof. Let n be a positive integer with $\varphi(n) = 6$ and r be a prime divisor of n . Then $r - 1$ divides 6, and hence $r = 2, 3$ or 7. If $r = 7$ then r^2 does not divide n , and hence $\varphi(n/7) = 1$, so $n = 7$ or $n = 14$. Therefore $r = 2$ or $r = 3$. If $n = r^\alpha$ for an integer $\alpha \geq 1$ then $n = 9$. It remains to consider the case $n = 2^\alpha 3^\beta$ for $\alpha, \beta \geq 1$. In this situation $\varphi(n) = 2^\alpha 3^{\beta-1}$. So $\alpha = 1, \beta = 2$, and hence $n = 18$. Since $6 = \varphi(7) = \varphi(14) = \varphi(9) = \varphi(18)$, the case $\varphi(n) = 6$ is done.

Let $\varphi(n) = 8$ and r be a prime divisor of n . Then $r - 1$ divides 8, and hence $r = 2, 3$ or 5. Let $n = r^\alpha$, where $\alpha \geq 1$ and $r \in \{2, 3, 5\}$. Then $r = 2$, and hence $n = 16$. If n is divisible by both 3 and 5 then $n_{\{3\}} = 3$ and $n_{\{5\}} = 5$. So $\varphi(n/15) = 1$, and hence $n \in \{15, 30\}$. Therefore either $n = 2^\alpha 3^\beta$ or $n = 2^\alpha 5^\beta$ for some integers $\alpha, \beta \geq 1$. Then $\beta = 1$ and either $\alpha = 3$ or $\alpha = 2$ respectively. So $n = 24$ or $n = 20$. Moreover, it is clear that if $n \in \{15, 16, 20, 24, 30\}$ then $\varphi(n) = 8$. The lemma is proved. \square

Lemma 2.7. *Let j and u be integers, $u \geq 3$, and either $j \geq 15$ or $\eta(j) \geq 9$. Then either $\eta(j) = 9$ or $k_j(u) \geq u^8/6$.*

Proof. Suppose at first that $j < 105$. By the lemma assumption, we have that $\varphi(j) \geq 6$. Let $\varphi(j) = 6$. Then $j \in \{7, 9, 14, 18\}$ due to Lemma 2.6. Since either $j \geq 15$ or $\eta(j) \geq 9$, we obtain that $\eta(j) = 9$. So we may assume that $\varphi(j) > 6$, and hence $\varphi(j) \geq 8$. Let r be the greatest prime divisor of j . Suppose that $\varphi(j) = 8$. By Lemma 2.6, we obtain that $j \in \{15, 16, 20, 24, 30\}$. If $j \neq 16, 20$ then $j_{\{r\}}$ does not divide $r - 1$, and hence $k_j(u) = \Phi_j(u)$ due to (2.1). Lemma 2.3 implies that $\Phi_j(u) \geq u^8/2 > u^8/6$, as required. If $j = 20$ then $k_j(u) = \frac{u^8 - u^6 + u^4 - u^2 + 1}{(5, u^2 + 1)} \geq (u^8 - u^6 + u^4 - u^2 + 1)/5 > u^8/6$. If $j = 16$ then $k_j(u) = (u^8 + 1)/(u - 1, 2) \geq u^8/2$. So we may assume that $\varphi(j) \geq 10$. Observe that $k_j(u) \geq \Phi_j(u)/r \geq u^{\varphi(j)}/(2r)$ due to Lemma 2.3. If $r \leq 17$ then $u^{\varphi(j)} \geq u^{10} \geq 9u^8$, and hence $k_j(u) \geq u^8/4$. It remains to treat the case $r > 17$. Since $\varphi(j)$ is divisible by $r - 1$, we have the inequality $\varphi(j) \geq r - 1$. Therefore $k_j(u) \geq u^{r-1}/(2r) \geq 3^{r-9}u^8/(2r)$. So it is sufficient to show that $3^{r-9}/(2r) \geq 1/6$ or equivalently $3^{r-8} \geq r$. This inequality is clearly true for $r = 19$, and hence it holds for all $r \geq 19$ because the next prime after r is less than $2r$.

Let now $j \geq 105$. We prove that $\varphi(j) \geq 16$. If j is divisible by rs , where r, s are prime greater than or equal to 5, then either $r \neq s$, and hence $(r - 1)(s - 1)$ divides $\varphi(j)$ or $r = s$, and hence $r(r - 1)$ divides $\varphi(j)$. It is clear that in the both cases $\varphi(j) \geq 16$. Let j has exactly one prime divisor r such that $r \geq 5$. If $r \geq 17$ then obviously $\varphi(j) \geq 16$. If $r \leq 13$ then $j/r > 8$, and hence $\varphi(j) = \varphi(r)\varphi(j/r) \geq 4 \cdot 4 = 16$. Thus we may assume that $j = 2^\alpha 3^\beta$, where α, β are integers. Observe that $\alpha, \beta \geq 1$, since otherwise the inequality $\varphi(j) \geq 16$ is clear. Now $\varphi(j) = 2^\alpha 3^{\beta-1}$. So if $\varphi(j) < 16$ then $\alpha < 4$ and $\beta < 3$. Whence $j \leq 8 \cdot 9 < 105$; a contradiction. Therefore we have that $\varphi(j) \geq 16$, and hence by Lemma 2.4 we obtain that $k_j(u) > u^{\varphi(j)/2} \geq u^8 > u^8/6$, as required. The lemma is proved. \square

3. Preliminaries: the prime graph and the spectrum of a finite group

Let G be a finite group. Observe that the spectrum of G is completely determined by the set $\mu(G)$ consisting of all maximal elements of $\omega(G)$ with respect to divisibility. The *prime graph* $GK(G)$ of G is defined as follows: its vertices are elements of $\pi(G)$, and two distinct vertices r and s are adjacent if and only if $rs \in \omega(G)$. Recall that a subset of vertices of a graph is called a *coclique*, if every two vertices of this subset are nonadjacent. Denote by $t(G)$ the greatest size of a coclique in $GK(G)$. We refer to a coclique containing r as an $\{r\}$ -coclique. If $r \in \pi(G)$ then $t(r, G)$ denotes the greatest size of $\{r\}$ -cocliques and $\rho(r, G)$ is a set of vertices in some $\{r\}$ -coclique of size $t(r, G)$.

Lemma 3.1. ([13, Proposition 2], [14, Theorem 2]) *Let L be a finite nonabelian simple group with $t(L) \geq 3$ and $t(2, L) \geq 2$, and let G be a finite group isospectral to L . Then the following hold.*

- (i) *There exists a nonabelian simple group S such that $S \leq \bar{G} = G/K \leq \text{Aut } S$ for the soluble radical K of G .*
- (ii) *For every coclique ρ of $GK(G)$ containing at least three elements, at most one prime from ρ divides the product $|K| \cdot |\bar{G}/S|$. In particular, $t(S) \geq t(L) - 1$.*
- (iii) *Every prime $r \in \pi(G)$ nonadjacent to 2 in $GK(G)$ does not divide $|K| \cdot |\bar{G}/S|$. In particular, $t(2, S) \geq t(2, L)$.*

Alongside with the function $\eta(k)$ (see (2.2)), we define several functions of natural argument that were used in [20] for formulation an adjacency criterion in the prime graph of a simple classical group.

$$(3.1) \quad \nu(k) = \begin{cases} k, & \text{if } k \equiv 0 \pmod{4}, \\ k/2, & \text{if } k \equiv 2 \pmod{4}, \\ 2k, & \text{if } k \text{ is odd.} \end{cases}$$

For $\varepsilon \in \{+, -\}$, put

$$(3.2) \quad \nu_\varepsilon(k) = \begin{cases} k, & \text{if } \varepsilon = +, \\ \nu(k) & \text{if } \varepsilon = -. \end{cases}$$

For linear and unitary groups, we exploit also a reformulation of an adjacency criterion (see [19, Lemmas 2.1–2.3]), if it is more convenient for our goals than an initial formulation from [20] which used the function ν_ε . This reformulation is based on the equality $k_{\nu_\varepsilon(i)}(q) = k_i(\varepsilon q)$, which follows from Lemma 2.2 and the definition of ν_ε .

Now we introduce a function φ which was defined in [12] in order to unify further arguments. Namely, given a simple classical group L over a field of order q and a prime r coprime to q , we put

$$(3.3) \quad \varphi(r, L) = \begin{cases} e(r, \varepsilon q), & \text{if } L = L_n^\varepsilon(q), \\ \eta(e(r, q)), & \text{if } L \text{ is symplectic or orthogonal.} \end{cases}$$

It follows that

$$(3.4) \quad e(r, q) = \begin{cases} 2\varphi(r, L), & \text{if either } e(r, q) \text{ is even and } L \text{ is symplectic or orthogonal,} \\ & \text{or } e(r, q) \equiv 2 \pmod{4} \text{ and } L \text{ is unitary;} \\ \varphi(r, L)/2, & \text{if } e(r, q) \equiv 1 \pmod{2} \text{ and } L \text{ is unitary;} \\ \varphi(r, L) & \text{otherwise.} \end{cases}$$

Observe that $e(r, -q) = \varphi(r, L)$ in the case of $e(r, q) = \varphi(r, L)/2$.

Following [12], for a classical group L , we write $\text{prk}(L)$ to denote its dimension if L is a linear or unitary group, and its Lie rank if L is a symplectic or orthogonal group. Observe that $n = \text{prk}(L)$ in Theorems 1.2, 1.3 in Introduction.

Lemma 3.2. *Let L be a finite simple group of Lie type over a field of characteristic p and $n = \text{prk}(L)$. Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, -q)$ and $l = e(s, -q)$ if $L \simeq L_n^-(q)$ and $k = e(r, q)$ and $l = e(s, q)$ otherwise. Suppose that $2 \leq \varphi(r, L) \leq \varphi(s, L)$. Then the following hold.*

(i) *If $L = L_n^\varepsilon(q)$ then r and s are nonadjacent in $GK(L)$ if and only if $\varphi(r, L) + \varphi(s, L) > n$, and $\frac{l}{k}$ is not a natural number.*

(ii) *If $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ then r and s are nonadjacent in $GK(L)$ if and only if $\varphi(r, L) + \varphi(s, L) > n$, and $\frac{l}{k}$ is not an odd natural number;*

(iii) *If $L = O_{2n}^\varepsilon(q)$ then r and s are nonadjacent in $GK(L)$ if and only if $2\varphi(r, L) + 2\varphi(s, L) > 2n - (1 - \varepsilon(-1)^{k+l})$, $\frac{l}{k}$ is not an odd natural number, and, if $\varepsilon = +$, then the chain of equalities: $n = l = 2\varphi(s, L) = 2\varphi(r, L) = 2k$ is not true.*

Proof. An adjacency criterion for two odd vertices in $GK(L)$ was obtained in [20, 21]. In particular, the statement (i) follows from [20, Propositions 2.1,2.2] and the statements (ii), (iii) follow from [21, Propositions 2.4,2.5]. □

Lemma 3.3. ([12, Lemma 2.3]) *Let L be a simple classical group over a field of order q and characteristic p . If r is an odd prime from $\pi(L) \setminus \{p\}$ then $\varphi(r, L)$ divides $r - 1$, and if L is a symplectic or orthogonal group then $2\varphi(r, L)$ divides $r - 1$.*

Lemma 3.4. ([12, Lemma 2.4]) *Let L be a simple classical group over a field of order q and characteristic p , and let $\text{prk}(L) = n \geq 4$.*

(i) *If $r \in \pi(L) \setminus \{p\}$, then $\varphi(r, L) \leq n$.*

(ii) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ with $\varphi(r, L) \leq n/2$ and $\varphi(s, L) \leq n/2$, then r and s are adjacent in $GK(L)$.*

(iii) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ with $n/2 < \varphi(r, L) \leq n$ and $n/2 < \varphi(s, L) \leq n$, then r and s are adjacent in $GK(L)$ if and only if $e(r, q) = e(s, q)$.*

(iv) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ and $e(r, q) = e(s, q)$, then r and s are adjacent in $GK(L)$.*

Let L be a simple classical group over a field of order q and characteristic p . For $\sigma \subseteq \pi(L) \setminus \{p\}$, put $E(\sigma, L) = \{e(r, q) \mid r \in \sigma\}$. If $\text{prk}(L) = n \geq 13$ then, by [21], every coclique ρ of greatest size in $GK(L)$ does not contain p , so the set $E(\rho, L)$ is well-defined for ρ . Define $J(L)$ as the union of sets $E(\rho, L)$, and $E(L)$ as the intersection of these sets, where ρ runs over all cocliques of greatest size in $GK(L)$.

Lemma 3.5. ([12, Lemma 2.5, Table 1]) *Let L be a simple classical group over a field of order q and characteristic p , and $\text{prk}(L) = n \geq 13$. Let ρ be a coclique of greatest size in $GK(L)$. If $J(L) = E(L)$ then*

$E(\rho, L) = E(L)$. If $J(L) \neq E(L)$ then $E(\rho, L) = E(L) \cup \{j\}$ for some $j \in J(L) \setminus E(L)$. In particular, $|E(L)| \leq t(L) \leq |E(L)| + 1$. The sets $E(L)$, $J(L) \setminus E(L)$ and numbers $t(L)$ are listed in Table 1.

TABLE 1. Cocliques of greatest size ($n \geq 13$)

| L | Conditions | $t(L)$ | $E(L)$ | $J(L) \setminus E(L)$ |
|---------------------------------|-----------------------|------------------|---|-------------------------------|
| $L_n^\varepsilon(q)$ | n odd | $\frac{n+1}{2}$ | $\{i \mid \frac{n}{2} < \nu_\varepsilon(i) \leq n\}$ | \emptyset |
| | n even | $\frac{n}{2}$ | $\{i \mid \frac{n}{2} < \nu_\varepsilon(i) < n\}$ | $\{\frac{n}{2}, n\}$ |
| $S_{2n}(q)$ or $O_{2n+1}(q)$ | $n \equiv 0 \pmod{4}$ | $\frac{3n+4}{4}$ | $\{i \mid \frac{n}{2} \leq \eta(i) \leq n\}$ | \emptyset |
| | $n \equiv 1 \pmod{4}$ | $\frac{3n+5}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n\}$ | \emptyset |
| | $n \equiv 2 \pmod{4}$ | $\frac{3n+2}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n\}$ | $\{\frac{n}{2}, n\}$ |
| | $n \equiv 3 \pmod{4}$ | $\frac{3n+3}{4}$ | $\{i \mid \frac{n+1}{2} < \eta(i) \leq n\}$ | $\{\frac{n-1}{2}, n-1, n+1\}$ |
| $O_{2n}^+(q)$ | $n \equiv 0 \pmod{4}$ | $\frac{3n}{4}$ | $\{i \mid \frac{n}{2} \leq \eta(i) \leq n, i \neq 2n\}$ | \emptyset |
| | $n \equiv 1 \pmod{4}$ | $\frac{3n+1}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq 2n, n+1\}$ | $\{n-1, n+1\}$ |
| | $n \equiv 2 \pmod{4}$ | $\frac{3n-2}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq 2n\}$ | $\{\frac{n}{2}, n\}$ |
| | $n \equiv 3 \pmod{4}$ | $\frac{3n+3}{4}$ | $\{i \mid \frac{n-1}{2} \leq \eta(i) \leq n, i \neq 2n, n-1\}$ | \emptyset |
| $O_{2n}^-(q)$ | $n \equiv 0 \pmod{4}$ | $\frac{3n+4}{4}$ | $\{i \mid \frac{n}{2} \leq \eta(i) \leq n\}$ | \emptyset |
| | $n \equiv 1 \pmod{4}$ | $\frac{3n+1}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq n, \frac{n+1}{2}\}$ | $\{\frac{n+1}{2}, n-1\}$ |
| | $n \equiv 2 \pmod{4}$ | $\frac{3n+2}{4}$ | $\{i \mid \frac{n}{2} < \eta(i) \leq n\}$ | $\{\frac{n}{2}, n-2, n\}$ |
| | $n \equiv 3 \pmod{4}$ | $\frac{3n+3}{4}$ | $\{i \mid \frac{n-1}{2} \leq \eta(i) \leq n, i \neq n, \frac{n-1}{2}\}$ | \emptyset |

Following [12], we call a prime $r \in \pi(L)$ *large* (with respect to L), if r lies in some coclique of greatest size in the prime graph $GK(L)$, and *small* (with respect to L) otherwise.

Lemma 3.6. ([12, Lemma 2.7]) *Let L be a simple classical group and $n = \text{prk}(L) \geq 13$.*

- (i) *If $\varphi(r, L) \geq n/2$, then r is large with respect to L .*
- (ii) *If r is large with respect to L , then $\varphi(r, L) \geq n/2 - 1$.*
- (iii) *If ρ is a coclique in $GK(L)$ and $n/2 < \varphi(r, L)$ for every $r \in \rho$, then $GK(L)$ has a coclique σ of size $t(L)$ with $\rho \subseteq \sigma$.*

Lemma 3.7. ([12, Lemma 2.8]) *Let L be a simple classical group over a field of order q and $n = \text{prk}(L) \geq 13$. Suppose that $r \in \pi(L)$ and ρ is an $\{r\}$ -coclique of greatest size in $GK(L)$. Then every $s \in \rho' = \rho \setminus \{r\}$ is large with respect to L . Further, if r is small with respect to L , then $\varphi(s, L) > n/2$ for every $s \in \rho'$, and $E(\rho', L)$ is uniquely determined by r . If, in addition, $i = e(r, q) > 2$, then $E(\rho', L)$ is uniquely determined by i .*

Let L be a simple classical group and $n = \text{prk}(L) \geq 13$. Let r be small with respect to L and ρ be an $\{r\}$ -coclique of greatest size. By Lemma 3.7, the set $E(\rho \setminus \{r\}, L)$ is contained in $J(L)$ and does not depend on a choice of ρ , so we denote it by $J(r, L)$.

It follows from Lemma 3.1 that primes nonadjacent with 2 in $GK(L)$ are important for L . Let

$$m(L) = \{n \in \mu(L) \mid \text{there exists } r \in \pi(n) \text{ such that } 2 \text{ and } r \text{ are nonadjacent in } GK(L)\}.$$

Lemma 3.8. *Let L be a simple classical group over a field of order q and $n = \text{prk}(L) \geq 13$.*

- (i) *The set $m(L)$ is as in Table 2, in particular, this set is nonempty.*
- (ii) *If $r \in \pi(L)$ is nonadjacent with 2 in $GK(L)$ then there exists a unique number $x \in m(L)$ such that r divides x .*

- (iii) *If $L = L_n^\varepsilon(q)$ and $x \in m(L)$ then $\frac{q^{n-1}-\varepsilon 1}{2(n, q-\varepsilon 1)} \leq x \leq \frac{2q^{n-1}}{(n, q-\varepsilon 1)}$.*
- (iv) *If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $x \in m(L)$ then $\frac{q^n-1}{(2, q-1)} \leq x \leq \frac{q^n+1}{(2, q-1)}$.*
- (v) *If $L = O_{2n}^+(q)$, n is even, and $x \in m(L)$, then $\frac{q^{n-1}}{2(2, q-1)} \leq x \leq \frac{2q^{n-1}}{(2, q-1)}$.*
- (vi) *If $L = O_{2n}^-(q)$, n is even, and $x \in m(L)$, then $\frac{q^n}{2(2, q-1)} \leq x \leq \frac{2q^n}{(2, q-1)}$.*
- (vii) *If $L = O_{2n}^\varepsilon(q)$, n is odd, and $x \in m(L)$, then $\frac{q^n}{2(4, q-\varepsilon 1)} \leq x \leq \frac{2q^n}{(4, q-\varepsilon 1)}$.*

Proof. It follows from [20, Propositions 6.3, 6.6], that if $r \in \pi(L)$ is nonadjacent with 2 in $GK(L)$ then there exists a maximal $\{2\}$ -coclique $\rho(2, L)$ containing r . A description of sets $\rho(2, L)$ for a finite simple classical group L is contained in [20, Tables 4, 6]. Moreover, information from these tables implies that if r is nonadjacent with 2 then r is nonadjacent with the characteristic of the underlying field of L . So orders in $\mu(L)$ not divisible by r are orders of semisimple elements of L . Spectrum descriptions of simple classical groups are contained in [1, 2], in particular, for every r nonadjacent with 2 in $GK(L)$, there exists exactly one element in $\mu(L)$ dividing on r . We combine mentioned descriptions of $\rho(2, L)$ and $m(L)$ into Table 2. Further in this proof we use the description of $m(L)$ from Table 2. Denote by x an element from $m(L)$.

Let $L = L_n^\varepsilon(q)$. If $\varepsilon = +$ then $\frac{q^{n-1}-1}{(n, q-1)} \leq x \leq \frac{q^n-1}{(q-1)(n, q-1)}$. So $x > \frac{q^{n-1}-1}{2(n, q-1)}$. Moreover, $\frac{q^n-1}{(q-1)(n, q-1)} = \frac{q^{n-1}+q^{n-2}+\dots+1}{(n, q-1)} \leq \frac{2q^{n-1}}{(n, q-1)}$, and hence the required inequalities hold for x in this case. If $\varepsilon = -$ then $\frac{q^n-(-1)^n}{(q+1)(n, q+1)} \leq x \leq \frac{q^{n-1}-(-1)^{n-1}}{(n, q+1)}$. Obviously, we have that $\frac{q^{n-1}-(-1)^{n-1}}{(n, q+1)} \leq \frac{2q^{n-1}}{(n, q+1)}$. It remains to check that $\frac{q^n-(-1)^n}{(q+1)(n, q+1)} \geq \frac{q^{n-1}+1}{2(n, q+1)}$. This is equivalent to $2(q^n - (-1)^n) \geq (q^{n-1} + 1)(q + 1)$ which is obviously true due to $n \geq 13$.

Now we prove statements (iv) and (v). If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ then $x = \frac{q^n \pm 1}{(2, q-1)}$. If $L = O_{2n}^+(q)$ with even n then $x = \frac{q^{n-1} \pm 1}{(2, q-1)}$. Obviously required inequalities hold in these cases.

Let $L = O_{2n}^-(q)$, where n is even. If q is odd then $x = (q^n + 1)/2$ and $q^n/4 < x < q^n$, as required. If q is even then $(q^{n-1} + 1)(q - 1) \leq x \leq (q^{n-1} - 1)(q + 1)$. Since $(q^{n-1} + 1)(q - 1) = q^n - q^{n-1} + q - 1 > q^n/2$ and $(q^{n-1} - 1)(q + 1) = q^n + q^{n-1} - q + 1 < 2q^n$, we obtain required inequalities for x .

Let finally $L = O_{2n}^\varepsilon(q)$ and n is odd. If $\varepsilon = +$ then we have that $\frac{q^n-1}{(4, q-1)} \leq x \leq \frac{(q^{n-1}+1)(q+1)}{(4, q-1)}$ and the inequalities for x hold due to $(q^{n-1} + 1)(q + 1) \leq 2q^n$ and $q^n - 1 > q^n/2$. If $\varepsilon = -$ then $\frac{(q^{n-1}+1)(q-1)}{(4, q+1)} \leq x \leq \frac{q^n+1}{(4, q+1)}$ and the inequalities for x hold due to $(q^{n-1} + 1)(q - 1) > q^n/2$ and $q^n + 1 < 2q^n$. The lemma is proved. □

TABLE 2. The set $m(L)$, $\text{prk}(L) \geq 13$

| S | Conditions | $\rho(2, S) \setminus \{2\}$ | The elements of $m(L)$ |
|---------------------------------------|--|--|---|
| $L_n^\varepsilon(q)$ | $n_{\{2\}} < (q - \varepsilon 1)_{\{2\}}$ | $\{r_n(\varepsilon q)\}$ | $\frac{q^n - (\varepsilon 1)^n}{(q - \varepsilon 1)(q - \varepsilon 1, n)}$ |
| | either $n_{\{2\}} > (q - \varepsilon 1)_{\{2\}} > 1$ or $n_{\{2\}} = (q - \varepsilon 1)_{\{2\}} = 2$ | $\{r_{n-1}(\varepsilon q)\}$ | $\frac{q^{n-1} - (\varepsilon 1)^{n-1}}{(q - \varepsilon 1, n)}$ |
| | either $q_{\{2\}} > 1$ or $2 < n_{\{2\}} = (q - \varepsilon 1)_{\{2\}}$ | $\{r_{n-1}(\varepsilon q), r_n(\varepsilon q)\}$ | $\frac{q^{n-1} - (\varepsilon 1)^{n-1}}{(q - \varepsilon 1, n)}, \frac{q^n - (\varepsilon 1)^n}{(q - \varepsilon 1)(q - \varepsilon 1, n)}$ |
| $O_{2n+1}(q)$ or $S_{2n}(q)$ | n is odd and $(q - 1)_{\{2\}} = 2$ | $\{r_n(q)\}$ | $\frac{q^n - 1}{2}$ |
| | n is even or $(q - 1)_{\{2\}} > 2$ | $\{r_{2n}(q)\}$ | $\frac{q^n + 1}{(2, q - 1)}$ |
| | n is odd and q is even | $\{r_n(q), r_{2n}(q)\}$ | $q^n - 1, q^n + 1$ |
| $O_{2n}^\varepsilon(q)$ n is odd | $q \equiv -\varepsilon 1 \pmod{4}$ | $\{r_n(\varepsilon q)\}$ | $\frac{q^n - \varepsilon 1}{2}$ |
| | $q - \varepsilon 1 \equiv 0 \pmod{8}$ | $\{r_{2n-2}(q)\}$ | $\frac{(q^{n-1} + 1)(q + \varepsilon 1)}{4}$ |
| | $q - \varepsilon 1 \equiv 4 \pmod{8}$ | $\{r_n(\varepsilon q), r_{2n-2}(q)\}$ | $\frac{q^n - \varepsilon 1}{4}, \frac{(q^{n-1} + 1)(q + \varepsilon 1)}{4}$ |
| | q is even | $\{r_n(\varepsilon q), r_{2n-2}(q)\}$ | $q^n - \varepsilon 1, (q^{n-1} + 1)(q + \varepsilon 1)$ |
| $O_{2n}^+(q)$ n is even | $q \equiv \tau 1 \pmod{4}, \tau \in \{+, -\}$ | $\{r_{n-1}(-\tau q)\}$ | $\frac{q^{n-1} + \tau 1}{2}$ |
| | q is even | $\{r_{n-1}(q), r_{2n-2}(q)\}$ | $q^{n-1} - 1, q^{n-1} + 1$ |
| $O_{2n}^-(q)$ n is even | q is odd | $\{r_{2n}(q)\}$ | $\frac{q^n + 1}{2}$ |
| | q is even | $\{r_{n-1}(q), r_{2n-2}(q), r_{2n}(q)\}$ | $(q^{n-1} - 1)(q + 1), (q^{n-1} + 1)(q - 1), q^n + 1$ |

Lemma 3.9. Let L, S be finite simple classical groups such that $\text{prk}(L) \geq 13, \text{prk}(S) \geq 13$. Assume that G is a finite group isospectral to L and $S \leq G/K \leq \text{Aut } S$, where K is the soluble radical of G . Then for every $m_1 \in m(L)$ there exists $m_2 \in m(S)$ such that m_2 divides m_1 .

Proof. Let $m_1 \in m(L)$. Then there exists r such that r is nonadjacent with 2 in $GK(L)$ and r divides m_1 . Lemma 3.1 implies that $r \in \pi(S)$, and r is nonadjacent with 2 in $GK(S)$. Therefore there exists $m_2 \in m(S)$ dividing by r . Applying Lemma 3.8 (ii), we obtain that m_2 divides m_1 , as required. \square

Lemma 3.10. Let L be a simple classical group over a field of order q and characteristic p .

- (i) If $L = L_n^\varepsilon(q)$ and $n \geq 23$, then $\omega(L)$ contains a number k with $k \geq q^{4t(L)/3}$ and all prime divisors of k are large with respect to L .
- (ii) If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $n \geq 17$, or $L = O_{2n}^\varepsilon(q)$ and $n \geq 18$, then $\omega(L)$ contains a number k with $k \geq q^{t(L)}/2$ and all prime divisors of k are large with respect to L .
- (iii) If $\text{prk}(L) \geq 14$ then the numbers from $\omega(L)$ do not exceed $q^{2t(L)}/(q - 1)$.

Proof. The statement (i) is exactly [12, Lemma 2.14 (i)]. It is also proved in [12, Lemma 2.14 (ii)] that if $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $n \geq 29$, or $L = O_{2n}^\varepsilon(q)$ and $n \geq 30$, then $\omega(L)$ contains a number k with $k \geq q^{10t(L)/9}$ and all prime divisors of k are large with respect to L . So we may assume that if $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ then $n \leq 28$, and if $L = O_{2n}^\varepsilon(q)$ then $n \leq 29$. It is easy to see that under these assumptions there exists j such that $r_j(q)$ is large with respect to L , j is prime, $t(L) < j \leq \text{prk}(L)$, and $j \neq \text{prk}(L)$ if $L = O_{2n}^\varepsilon(q)$. Then $r_{2j}(q)$ is also large with respect to L . Remind that $r_{2j}(q) = r_j(-q)$ and choose $\varepsilon \in \{+, -\}$ such that $(q - \varepsilon 1, j) = 1$. Now we prove that $k = k_j(\varepsilon q)$ is required. It is clear that all prime divisor of k are large with respect to L , so it is sufficient to show that $k \geq q^{t(L)}/2$. Since $(q - \varepsilon 1, j) = 1$, we have that $k = (q^j - \varepsilon 1)/(q - \varepsilon 1) = (\varepsilon q)^{j-1} + (\varepsilon q)^{j-2} + \dots + \varepsilon q + 1$. Since $q \geq 2$, we have that $q^{j-1} \geq 2q^{j-2}$, and hence $k > q^{j-1}/2$. Since we choose j so that $j > t(L)$, we obtain the required inequality $k > q^{t(L)}/2$.

Now we prove (iii). It follows from [17, Lemma 1.3] that elements of $\omega(L)$ do not exceed $q^{m+1}/(q-1)$, where m is the Lie rank of L . If L is linear or unitary then by Table 1, we obtain that $t(L) \geq \text{prk}(L)/2 = (m+1)/2$, and hence elements of $\omega(L)$ do not exceed $q^{2t(L)}/(q-1)$, as claimed. If L is symplectic or orthogonal then by Table 1 we have that $t(L) \geq (3m-2)/4$. So it is sufficient to prove that $(3m-2)/2 \geq m+1$. This is equivalent to $m \geq 4$. The lemma is proved. \square

Lemma 3.11. ([12, Lemma 2.12]) *Let L be a simple classical group over a field of order q and characteristic p , and let $\text{prk}(L) = n \geq 4$. If $r \in \pi(L) \setminus \{p\}$, $i = e(r, q)$, and $n/2 < \varphi(r, L) \leq n$, then L includes a cyclic Hall subgroup of order $k_i(q)$.*

4. Proof: restrictions on K and \overline{G}/S

Let \mathcal{L} be the class of simple classical groups whose dimension are as in the hypothesis of Theorem 1.3. It is not hard to check using information in Table 1 that $L \in \mathcal{L}$ if and only if $14 \leq t(L) \leq 22$.

Let L be a simple classical group over a field of order q and characteristic p and $L \in \mathcal{L}$. Suppose that G is a finite group isospectral to L . By [20, Theorem 7.1], we have that $t(2, L) \geq 2$. Lemma 3.1 implies that there exists a nonabelian simple group S such that $S \leq G/K \leq \text{Aut } S$, where K is the soluble radical of G . Further in this section we fix groups G , K , and S . Moreover, we suppose that S is a simple classical group over a field of order u and characteristic v .

The following three propositions are the analogies of the [12, Propositions 3, 4, 5]. There are only two differences. First, we require $L \in \mathcal{L}$ instead of $\dim L \geq 40$ and, second, there are no exceptions in Proposition 4.2 in contrast to [12, Proposition 4], where there are some additional restrictions in cases of linear and unitary groups.

Proposition 4.1. *Suppose that $L \in \mathcal{L}$ and q is odd. Then the soluble radical K of G is nilpotent. If $r \in \pi(K) \setminus \{v\}$, then $t(r, L) = 2$, and $(s, |K| \cdot |\overline{G}/S| \cdot |P|) = 1$ for every $s \in \pi(L)$ nonadjacent to r in $GK(L)$ and every proper parabolic subgroup P of S .*

Proposition 4.2. *Suppose that $L \in \mathcal{L}$ and q is odd. If a prime r not equal to p divides the order of \overline{G}/S , then $\varphi(r, L) \leq n/3$. In particular, r is small with respect to L .*

Proposition 4.3. *Suppose that $L \in \mathcal{L}$ and q is odd. If a prime r is large with respect to L , then $(r, pv|K| \cdot |\overline{G}/S|) = 1$ and $k_{e(r,q)}(q) \in \omega(S)$. In particular, $t(S) \geq t(L)$.*

Proposition 4.2 will follow from Lemma 4.5 and Lemma 3.6. Proposition 4.1 is proved as [12, Proposition 3] with Lemma 4.5 in place of [12, Lemma 4.1]. The proof of [12, Proposition 5] uses [12, Proposition 4], and the condition $\dim L \geq 40$ is used to conclude that p is not large with respect to L . Actually it is sufficient to have that $t(L) \geq 6$, which is true since $L \in \mathcal{L}$. So the proof of Proposition 4.3 goes as the proof of [12, Proposition 5] with the changes pointed above.

Lemma 4.4. *Let $L \in \mathcal{L}$, q is odd, $r \in \pi(L)$, and r divides $|\overline{G}/S|$. Suppose that there exist $i, j, i \neq j$ such that $\{r, r_i(q), r_j(q)\}$ is a coclique in $GK(L)$. Then r divides either $k_i(q) - 1$ or $k_j(q) - 1$.*

Proof. Put $m = \text{prk}(S)$. Since r divides $|\overline{G}/S|$, Lemma 3.1(ii) implies that $(r_i \cdot r_j, |\overline{G}/S| \cdot |K|) = 1$ and $r_i, r_j \in \pi(S)$. Assume that $\varphi(r_i, S) \leq m/2$. By Lemma 3.4(ii), for all $r_j \in R_j(q)$ we obtain that $\varphi(r_j, S) > m/2$. So $e(r_j, S)$ is the same for all $r_j \in R_j(q)$ due to Lemma 3.4(iii). Put $t = e(r_j, u)$ and observe that S has

a cyclic Hall subgroup of order $k_t(u)$ due to Lemma 3.11. Let $s \in \pi(k_j(q))$ and $P \in Syl_s(S)$. By Frattini argument, r divides $|N_{\overline{G}}(P)|$. Let $x \in N_{\overline{G}}(P)$ and $|x| = r$. Then $S\langle x \rangle$ is a Frobenius group with kernel S and complement $\langle x \rangle$, and hence $|S| \equiv 1 \pmod{r}$. Note that Sylow s -subgroup of G and S are isomorphic and since s is an arbitrary element of $\pi(k_j(q))$ we obtain that $k_j(q) - 1$ is divisible by r . The case $\varphi(r_i, S) > m/2$ is exactly the same. □

Lemma 4.5. *Let $L \in \mathcal{L}$ and q is odd. If $r \in \pi(L) \setminus \{p\}$ and $\varphi(r, L) > n/3$, then r does not divide $|\overline{G}/S|$.*

Proof. Assume the contrary and let there exists r such that r divides $|\overline{G}/S|$ and $\varphi(r, L) > n/3$. First, we consider the case $L \simeq L_n^\varepsilon(q)$, $n \geq 27$. Put $i = e(r, \varepsilon q) = \varphi(r, L)$ and so $i > n/3$. Let M be the set from Table 3 which depends on n and i . The set consists of two elements that we denote by i_1, i_2 . It is easy to see that $n/2 < i_1, i_2 \leq n$, so $r_{i_1}(\varepsilon q), r_{i_2}(\varepsilon q)$ are large with respect to L by Lemma 3.6(i). Moreover, we have that $i_1 + i > n, i_2 + i > n$ and i_1, i_2 are not divisible by i . Lemma 3.2 implies that $r_{i_1}(\varepsilon q)$ and $r_{i_2}(\varepsilon q)$ are nonadjacent with r in $GK(L)$, so $\{r, r_{i_1}(\varepsilon q), r_{i_2}(\varepsilon q)\}$ is a coclique of size 3 in this graph. By Lemma 4.4, there exists $j \in M$ such that r divides $k_j(\varepsilon q) - 1$. Now we prove that it is impossible.

TABLE 3. Set M for $L_n^\varepsilon(q)$

| $n = 27$ | | $28 \leq n \leq 31$ | |
|---|----------|---|----------------------|
| i | M | i | M |
| 18, 27 | {16, 24} | $i \geq 14$ | see M for $n = 27$ |
| 12, 24, $\varepsilon q \equiv -1 \pmod{3}$ | {16, 27} | $i = 10, 11, 13$ | {24, 28} |
| 12, 24, $\varepsilon q \not\equiv -1 \pmod{3}$ | {16, 18} | $i = 12, \varepsilon q \equiv 1 \pmod{5}$ | {20, 27} |
| $i \neq 12, 18, 24, 27, \varepsilon q \equiv -1 \pmod{3}$ | {24, 27} | $i = 12, \varepsilon q \not\equiv 1 \pmod{5}$ | {25, 27} |
| $i \neq 12, 18, 24, 27, \varepsilon q \not\equiv -1 \pmod{3}$ | {18, 24} | | |
| $32 \leq n \leq 39$ | | $40 \leq n \leq 44$ | |
| i | M | i | M |
| $i \neq 11, 12, 14, 16, 28, 32$ | {28, 32} | $i \neq 16, 18, 20, 36, 40$ | {36, 40} |
| $i = 16, 32$ | {24, 28} | 20, 40 | {32, 36} |
| $i = 28$ | {24, 32} | 18, 36 | {32, 40} |
| $i = 12$ | {30, 32} | $i = 16, n < 44$ | {28, 36} |
| $i = 14, n \leq 37$ | {24, 32} | $i = 16, n = 44$ | {36, 44} |
| $i = 11, n = 32$ | {28, 32} | | |
| $i = 14, n \geq 38$ | {32, 36} | | |

Assume that $n = 27$. Then $i \geq 10$. By the definition of M , we have that $j \in \{16, 18, 24, 27\}$. If $j = 16$ then $k_j(\varepsilon q) - 1 = (q^8 - 1)/2$, and hence $i \leq 8$; a contradiction. If $j = 24$ then $k_j(\varepsilon q) - 1 = q^4(q^4 - 1)$, and hence $i \leq 4$; a contradiction. If $j = 18$ then by Table 3, we have that $\varepsilon q \not\equiv -1 \pmod{3}$, and hence $k_j(\varepsilon q) - 1 = \varepsilon q^3(\varepsilon q^3 - 1)$, $i \leq 3$; a contradiction. Finally, if $j = 27$ then $\varepsilon q \equiv -1 \pmod{3}$, $k_j(\varepsilon q) - 1 = \varepsilon q^9(\varepsilon q^9 + 1)$, and hence $i = 18$; a contradiction with the choice of M .

Assume that $28 \leq n \leq 31$, and hence $i \geq 10$. If $i \geq 14$ then M is the same as in the previous case, so this case is considered above. Let now $i < 14$ and $i \neq 12$. By the definition of M , we have that $j \in \{24, 28\}$. The

case $j = 24$ is regarded as for $n = 27$. If $j = 28$ then r divides $k_{28}(\varepsilon q) - 1 = q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2$, and hence r divides $q^{10} - q^8 + q^6 - q^4 + q^2 - 1 = (q^2 - 1)(q^8 + q^4 + 1)$. Therefore r divides $q^8 + q^4 + 1$, and hence $q^{12} - 1$ is divisible by r ; a contradiction. So $i = 12$ and r divides $k_j(\varepsilon q) - 1$ where $j = 20, 25$ or 27 . If $j = 20$ then $\varepsilon q \equiv 1 \pmod{5}$, and hence $k_{20}(\varepsilon q) - 1 = q^8 - q^6 + q^4 - q^2 = q^2(q^2 - 1)(q^4 + 1)$. So r divides either $q^2 - 1$ or $q^4 + 1$; a contradiction since $e(r, \varepsilon q) = 12$. Let $j = 25$ then $\varepsilon q \not\equiv 1 \pmod{5}$, and hence $k_{25}(\varepsilon q) - 1 = \varepsilon q^5(\varepsilon q^5 + 1)(q^{10} + 1)$. So r divides either $\varepsilon q^5 + 1$ or $q^{10} + 1$; a contradiction. Thus $j = 27$. In this case r divides $k_{27}(\varepsilon q) - 1 = \frac{q^{18} + \varepsilon q^9 + 1}{(q - \varepsilon 1, 3)} - 1$. If $(q - \varepsilon 1, 3) = 1$ then r divides $q^9(\varepsilon q^9 + 1)$; a contradiction. Let $(q - \varepsilon 1, 3) = 3$ then $k_{27}(\varepsilon q) - 1 = (q^9 - \varepsilon 1)(q^9 + \varepsilon 2)/3$. Since $e(r, \varepsilon q) = 12$, we have that $q^9 \equiv -\varepsilon 2 \pmod{r}$. Then $1 \equiv q^{12} \equiv -\varepsilon 2q^3 \pmod{r}$, and hence $1 \equiv -\varepsilon 8q^9 \equiv 16 \pmod{r}$. So r divides 15; a contradiction since $e(3, \varepsilon q), e(5, \varepsilon q) \leq 4$.

Assume that $32 \leq n \leq 39$, and hence $i \geq 11$. Observe that $i = 11$ only when $n = 32$. By the definition of M , we have that $j \in \{24, 28, 30, 32, 36\}$. If $j = 24$ then r divides $k_j(\varepsilon q) - 1 = q^4(q^4 - 1)$; a contradiction. If $j = 28$ then $k_j(\varepsilon q) - 1 = q^2(q^3 - 1)(q^3 + 1)(q^4 - q^2 + 1)$. Since $(q^4 - q^2 + 1) \mid (q^{12} - 1)$ and $i \neq 12$, we obtain a contradiction. If $j = 32$ then by definition of M we have $i \neq 16$ and r divides $k_j(\varepsilon q) - 1 = (q^{16} - 1)/2$; a contradiction. If $j = 36$ then $r_i(\varepsilon q)$ divides $k_j(\varepsilon q) - 1 = q^6(q^3 - 1)(q^3 + 1)$; a contradiction. Thus $j = 30$, and hence $i = 12$ by Table 3. Since $k_{30}(\varepsilon q) - 1 = \varepsilon q(q^4 - 1)(\varepsilon q^3 + q^2 - 1)$, we have that r divides $\varepsilon q^3 + q^2 - 1$. However, r divides $k_{12}(\varepsilon q) = q^4 - q^2 + 1$. So r divides $\varepsilon q(\varepsilon q^3 + q^2 - 1) - (q^4 - q^2 + 1) = \varepsilon q^3 + q^2 - 1 - \varepsilon q$. Therefore r divides εq ; a contradiction.

Finally we assume that $40 \leq n \leq 44$, and hence $i \geq 14$. By the definition of M , we have that $j \in \{28, 32, 36, 40, 44\}$. If $j = 28$ then $k_j(\varepsilon q) - 1 = q^2(q^3 - 1)(q^3 + 1)(q^4 - q^2 + 1)$. We get a contradiction, since $i \geq 14$. If $j = 32$ then by Table 3, we have that $i \neq 16$. Then r divides $k_j(\varepsilon q) - 1 = (q^{16} - 1)/2$; a contradiction. If $j = 36$ then $r_i(\varepsilon q)$ divides $k_j(\varepsilon q) - 1 = q^6(q^3 - 1)(q^3 + 1)$; a contradiction. If $j = 40$ then by Table 3, $j \neq 16$ and r divides $k_{40}(\varepsilon q) - 1 = q^4(q^4 - 1)(q^8 + 1)$; a contradiction. Thus $j = 44$, and hence $i = 16$. Then $k_j(\varepsilon q) - 1 = q^2(q^5 + 1)(q^5 - 1)(q^8 - q^6 + q^4 - q^2 + 1)$; a contradiction, since $(q^8 - q^6 + q^4 - q^2 + 1) \mid (q^{10} + 1)$. Thus L is not a linear or unitary group.

Assume that L is a symplectic or orthogonal group. We have the similar proof as in the case of linear and unitary groups. Put $i = e(r, q)$. Then $\varphi(r, L) = \eta(i)$ (see (3.3)). Let M be the set from Table 4 which depends on n, i . We have the similar properties of M as early: M consists of two numbers i_1, i_2 such that $\varphi(r_{i_1}(q), L) = \eta(i_1) > n/2$, $\varphi(r_{i_2}(q), L) = \eta(i_2) > n/2$, $\varphi(r_i(q), L) + \varphi(r_{i_1}(q), L) > n$, $\varphi(r_i(q), L) + \varphi(r_{i_2}(q), L) > n$ and $i_1/i, i_2/i$ are not odd integers. So $\{r_i, r_{i_1}, r_{i_2}\}$ is a coclique in $GK(L)$. Lemma 4.4 implies that there exists $j \in M$ such that r divides $k_j(\varepsilon q) - 1$. Now we prove that it is impossible sorting out values of n . By the assumption of Theorem 1.3, we have that $17 \leq n \leq 30$. Assume at first that $n = 17$ or $n = 18$. Then $\eta(i) \geq 6$ or $\eta(i) \geq 7$ respectively.

TABLE 4. Set M for orthogonal and symplectic groups

| $n = 17, 18$ | | $19 \leq n \leq 20$ | |
|---------------------------------|--------------|-------------------------------------|--------------|
| i | M | i | M |
| $i \neq 16, 24, 32$ | $\{24, 32\}$ | $i \neq 16, 32, 36$ | $\{32, 36\}$ |
| 16 | $\{24, 28\}$ | 16, 32 | $\{28, 36\}$ |
| 24 | $\{28, 32\}$ | 36 | $\{28, 32\}$ |
| 32 | $\{24, 28\}$ | | |
| $21 \leq n \leq 25$ | | $26 \leq n \leq 30$ | |
| i | M | i | M |
| $i \neq 16, 36, 40$ | $\{36, 40\}$ | $i \neq 20, 44, 48$ | $\{44, 48\}$ |
| 36 | $\{28, 40\}$ | $i = 48$ | $\{40, 44\}$ |
| 40 | $\{28, 36\}$ | $i = 44$ | $\{40, 48\}$ |
| $16, q \equiv -1 \pmod{19}$ | $\{36, 19\}$ | $i = 20, q \equiv -1 \pmod{23}$ | $\{23, 48\}$ |
| $16, q \not\equiv -1 \pmod{19}$ | $\{36, 38\}$ | $i = 20, q \not\equiv -1 \pmod{23}$ | $\{46, 48\}$ |

By Table 4, we obtain that $j \in \{24, 28, 32\}$. Assume that $j = 28$. Then $k_{28}(q) - 1 = q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2 = (q^4 - q^2)(q^8 + q^4 + 1)$. Hence r divides $q^8 + q^4 + 1$, whence $i = 12$; a contradiction with the choice of M . Let $j = 24$. Then r divides $q^8 - q^4 = q^4(q^4 - 1)$; a contradiction. If $j = 32$, then r divides $q^{16} - 1$. So $i = 16$; a contradiction with the choice of M .

Let now $19 \leq n \leq 20$. In this case $\eta(i) \geq 7$. The cases $j = 28, 32$ are treated in the same way as for $n = 18$. So $j = 36$, and hence $k_j(q) - 1 = q^{12} - q^6 = q^6(q^6 - 1)$; a contradiction with $\eta(i) \geq 7$.

Assume that $21 \leq n \leq 25$. So $\eta(i) \geq 8$. The cases $j = 28, 36$ are considered above. Let $j = 40$. Since $k_{40}(q) - 1 = q^4(q^4 - 1)(q^8 + 1)$, we have that $r \mid (q^8 + 1)$, and hence $i = 16$; a contradiction with the choice of M .

Let $j = 19$. Then $(q - 1, 19) = 1$, and hence $k_{19}(q) = \frac{q^{19}-1}{q-1}$. Since $i = 16$, we have r divides $q^{16} - 1$. Since r divides $k_{19}(q) - 1$, we have that $q^{19} \equiv q \pmod{r}$, and hence $q^3 \equiv q \pmod{r}$. So r divides $q(q^2 - 1)$; a contradiction. Thus $j = 38$ and $(q + 1, 19) = 1$. Therefore $k_{38}(q) - 1 = \frac{q^{19}+1}{q+1} - 1 = \frac{q^{19}-q}{q+1}$. Since $i = 16$, we have that $q^{16} \equiv 1 \pmod{r}$, and hence $q^{19} - q \equiv q^3 - q \pmod{r}$. So $q^3 - q$ is divisible by r ; a contradiction.

Assume that $26 \leq n \leq 30$, and hence $\eta(i) \geq 9$. Then $j \in \{23, 40, 44, 46, 48\}$. If $j = 40$ then similarly to the previous case we have that $i = 16$, however $\eta(16) = 8 < 9$; a contradiction. Let $j = 44$, and hence $i \neq 20$. Then $k_j(q) - 1 = q^2(q^{10} - 1)(q^8 - q^6 + q^4 - q^2 + 1)$. So $r \mid q^8 - q^6 + q^4 - q^2 + 1$, and hence $r \mid q^{10} + 1$; a contradiction; If $j = 48$ then $k_j(q) - 1 = q^8(q^8 - 1)$, so r divides $q^8 - 1$; a contradiction. Thus $j = 23$ or $j = 46$. In this case $i = 20$. Note that $k_j(q) = k_{23}(\tau q)$ for $\tau \in \{+, -\}$ and $q \not\equiv \tau 1 \pmod{23}$. Then $k_{23}(\tau q) - 1 = \frac{q^{23}-q}{q-\tau 1}$. So $q^{23} \equiv q \pmod{r}$ and since $q^{20} \equiv 1 \pmod{r}$, we obtain that $q^3 \equiv q \pmod{r}$. Therefore $r \mid (q^2 - 1)q$; a contradiction. The lemma is proved. \square

5. Proof: restrictions on S

Let L be as in the hypothesis of Theorem 1.3, that is, $L \in \mathcal{L}$ and the characteristic of underlying field of L is odd. As in the previous section, we assume further that G is a finite group isospectral to L , and S is a unique nonabelian composition factor of G , so $S \leq G/K \leq \text{Aut } S$, where K is the soluble radical of G .

Lemma 5.1. *S is a finite simple classical group.*

Proof. Since $L \in \mathcal{L}$, we have that $t(L) \geq 14$. By Lemma 3.1, we obtain that $t(S) \geq t(L) - 1 \geq 13$. By [19, Theorems 1, 2] and [18, Theorems 1, 2], S is neither an alternating group nor sporadic simple group nor the Tits group. By [21, Table 4], for any exceptional simple group of Lie type U , we have that $t(U) \leq 12$. So S is a simple classical group. \square

Further we assume that S is a simple classical group and numbers p, q, u, v are as in the previous section. Put $m = \text{prk}(S)$ and $t = t(L)$. Following to the hypothesis of Theorem 1.3, we assume that $2 \neq p \neq v$, in particular, $q \neq u$ and $q \geq 3$.

Proposition 5.2. *Suppose that $L \in \mathcal{L}$ and q is odd, and for some positive integer a the number $k_a(u)$ has a prime divisor large with respect to S . Then $k_a(u)$ has a prime divisor large with respect to L . In particular, $t(L) = t(S)$ and every prime r large with respect to L is large with respect to S .*

Proof. The assertion is an analogue of Proposition 7 in [12], where the conclusion is proved for L with $t(L) \geq 23$. We will proof Proposition 5.2 in the similar manner. Suppose to the contrary that non of prime divisors of $k_a(u)$ is large with respect to L .

Lemma 5.3. *For every coclique σ of greatest size in $GK(S)$ containing $r_a(u)$ there exists a set $J \subseteq E(\sigma, S)$ of size $d = \max\{1, t(S) - t(L)\}$ satisfying the following:*

- (i) *for any $j \in J$, every $r \in R_j(u)$ is large with respect to S and divides the number $|\overline{G}/S| \cdot |K|$;*
- (ii) *if $t(S) > t(L)$ and every coclique ρ of the greatest size in $GK(L)$ contains a prime s with $\varphi(s, S) \leq m/2$, then $\varphi(r, S) > m/2$ for any $j \in J$ and every $r \in R_j(u)$.*

Proof. The assertion is close to [12, Lemma 6.1]. One can check that [12, Lemma 6.1] states that there exists a set J of positive integers satisfying (i) and (ii). However, the proof starts from arbitrary coclique, say σ , of greatest size in $GK(S)$ containing $r_a(u)$ and the set J is proved to be a subset of $E(\sigma, S)$. Moreover, the proof is not used the condition $t(L) \geq 23$ from the mentioned above Proposition 7 from [12]. So the proof of the lemma goes as the proof of Lemma 6.1 in [12]. \square

Lemma 5.4. *Let a set J be as in Lemma 5.3. Then for each $j \in J$ and every prime r from $R_j(u)$ the number $(k_j(u))_{\{r\}}$ divides $p(q^2 - 1) \log_v u$. Moreover, the inequality*

$$\frac{\prod_{j \in J} k_j(u)}{\log_v u} \leq p(q^2 - 1)$$

holds true.

Proof. The statement is the exactly same as [12, Lemma 6.4]. The proof of Lemma 6.4 [12] uses [12, Propositions 3, 5] and auxiliary [12, Lemmas 6.2, 6.3]. One can check that Lemmas 6.2, 6.3 hold true for $L \in \mathcal{L}$. As was mentioned above, in place of Propositions 3, 5 from [12] we can use Propositions 4.1, 4.3. So the proof goes as the proof of [12, Lemma 6.4] with the pointed changes. \square

We are ready to prove Proposition 5.2. By Proposition 4.3, we conclude that $t(S) \geq t(L)$. Let J be a corresponding set from Lemma 5.3 and $d = |J|$. Then $d \geq t(S) - t$. By Lemma 5.3, we have that $J \subseteq E(\sigma, S)$, where σ is a coclique of greatest size in $GK(S)$.

Assume at first that $d \geq 2$ and $u \neq 2$. In this case $d = t(S) - t$, in particular $t(S) \geq 15$. By Table 1, we have that for any $j \in J$ either $j \geq 15$ or $\eta(j) \geq 9$. Lemma 2.7 implies that if $\eta(j) > 9$ and $u \geq 3$

then $k_j(u) \geq u^8/6$. If $\eta(j) = 9$ then $j \in \{9, 18\}$. Since $t(S) \geq 15$ and $J \subseteq E(\sigma, S)$, information from Table 1 implies that the both numbers 9 and 18 cannot be elements of J simultaneously. If $j \in \{9, 18\}$ then $k_j(u) \geq (u^6 - u^3 + 1)/3 \geq u^6/6$. So we obtain that $\prod_{j \in J} k_j(u) \geq u^6(u^8)^{d-1}/6^d = u^{8d-2}/6^d$. On the other hand, $\prod_{j \in J} k_j(u) \leq p(q^2 - 1) \log_v u < q^3 u$. Therefore $u^{8d-2}/6^d < q^3 u$, and hence $u^{8d-3}/6^d < q^3$. Since $u^2 \geq 9$, we have that $u^{8d-3} = u^{2d} u^{6d-3} \geq 9^d u^{6d-3} > 6^d u^{6d-3}$. So $u^{6d-3} < q^3$, and hence $u^{2d-1} < q$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L . It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq u^{2t(S)}/(u-1) \leq u^{2t+2d}/(u-1) \leq u^{2t+2d}/2$ by Lemma 3.10 (iii). Therefore $q^t \leq u^{2t+2d}$. Using the inequality $u^{2d-1} < q$, we obtain that $u^{(2d-1)t} < q^t \leq u^{2t+2d}$. To get a contradiction, it is sufficient to show that $(2d-1)t > 2t+2d$ or equivalently $2dt > 3t+2d$. Since $d \geq 2$, we have that $d \geq 3/2 + d/4$, and hence $2dt \geq 3t + td/4$. Now $3t + td/4 > 3t + 2d$ due to $t \geq 14$. Therefore $(2d-1)t > 2t+2d$, as claimed; a contradiction.

Let now $d = 1$ and $u > 2$. Then by Lemma 5.4, we have that $k_j(u)$ divides $p(q^2 - 1) \log_v u$, where $j \in J$. Observe that $p(q^2 - 1)$ is divisible by 24 and $k_j(u)$ is coprime to 6, so actually $24k_j(u)$ divides $p(q^2 - 1) \log_v u$. By Table 1, we have that either $j \geq 15$ or $\eta(j) \in \{7, 8, 9\}$. Assume at first that $j \geq 15$ and $\eta(j) \neq 7, 9$. Lemma 2.7 implies that $k_j(u) \geq u^8/6$. Since $8k_j(u)$ divides $p(q^2 - 1) \log_v u$ and $\log_v u \leq u$, we obtain that $8u^8/6 \leq q^3 u$, and hence $u^7 < q^3$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L . It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq u^{2t(S)}/(u-1) \leq u^{2t+2}/(u-1)$ by Lemma 3.10 (iii). So $q^t < 2u^{2t+2}/(u-1) < 4u^{2t+1}$. Since $u^7 < q^3$, we have that $u^{7t/3} < q^t$, and hence $u^{7t/3} < 4u^{2t+1}$. The last inequality is equivalent to $u^{t/3-1} < 4$; a contradiction, since $u \geq 3$ and $t \geq 14$. Therefore we have that $\eta(j) \in \{7, 9\}$. By Table 1, we have that either $t(S) \leq 15$ or $m \leq 36$ and S is a linear or unitary group in the latter case.

Assume that $t(L) = t(S)$. Then every coclique of greatest size in $GK(L)$ forms a coclique of greatest size in $GK(S)$ due to Proposition 4.3. Since $24 \in E(S)$, we have that there exists $r \in R_{24}(u)$ such that r is large with respect to L . Also we may assume that $r \in \sigma$. Let $r \in R_i(q)$. Now we prove that $R_i(q) \subseteq R_{24}(u)$. Let $r' \in R_i(q) \setminus \{r\}$. Then $r' \in \pi(S)$ and $t(r', S) = t$ due to Proposition 4.3. However r' is adjacent with r , and hence $r' \in R_{24}(u)$. So $R_i(q) \subseteq R_{24}(u)$, and hence $k_i(q)$ divides $k_{24}(u) = u^8 - u^4 + 1$. By Table 1, we have that either $\eta(i) \in \{7, 8, 9\}$ or $\eta(i) > 9$. In the first case we have that $k_i(q) \geq q^6/14$ and in the other cases by Lemma 2.7 we obtain that $k_i(q) \geq q^8/6$. Therefore $q^6 \leq 14u^8$, and hence $q^3 < 4u^4$. On the other hand, we have that $24k_j(u)$ divides $p(q^2 - 1) \log_v u$, and hence $24k_j(u) < q^3 u$. Since $\eta(j) \in \{7, 9\}$, we have that $k_j(u) \geq u^6/14$. Therefore $(24/14)u^6 \leq q^3 u$, and hence $u^5 < (3/4)q^3$. Comparing with the inequality $q^3 < 4u^4$, we obtain that $u^5 < 3u^4$; a contradiction with $u > 2$. So $t(S) - t(L) = 1$. Since $\eta(j) \in \{7, 9\}$ and $r_j(u)$ is large with respect to S , we obtain that either $S \simeq L_m^\pm(u)$ with $29 \leq m \leq 36$ or $S \in \{O_{38}^\pm(u), O_{39}(u), S_{38}(u)\}$. By Table 1, we may choose σ such that if $r \in \sigma$ and $\varphi(r, S) \leq m/2$ then $\varphi(r, S) = j$. Assume that there exists $i \in E(\sigma, S)$ such that for every $r \in R_i(u)$ it is true that r divides $|\overline{G}/S| \cdot |K|$. Then the set $\{i, j\}$ satisfies conditions of Lemma 5.3 and we obtain a contradiction as in the case $d \geq 2$ considering this set instead of J . Thus we can choose for every $i \in E(\sigma, S) \setminus \{j\}$ an element $r_i(u) \in R_i(u)$ such that $(r_i(u), |\overline{G}/S| \cdot |K|) = 1$. These $t(S) - 1$ elements form a coclique of greatest size in $GK(L)$, in particular $r_{24}(u)$ is large with respect to L , since $24 \in E(S)$. Let $r_{24}(u) \in R_i(q)$ and $r' \in R_i(q) \setminus \{r_{24}(u)\}$. Then $r' \in \pi(S)$, $t(r', S) \geq t(S) - 1$, and r' is adjacent to $r_{24}(u)$ in $GK(S)$. By the adjacency criterion, we obtain that $r' \in R_{24}(u)$. So $R_i(q) \subseteq R_{24}(u)$, and hence $k_i(q)$ divides $k_{24}(u)$. Arguing as above, we obtain a contradiction.

Let finally $u = 2$, in particular, $\log_u u = 1$. Then by Lemma 5.4, we have that $\prod_{j \in J} k_j(u)$ divides $p(q^2 - 1)$. Let r be a prime divisor of $\prod_{j \in J} k_j(u)$. Then r divides $k_j(u)$ for some $j \in J$, and hence $r - 1$ is divisible by j . Since for any $j \in J$, we have that $\eta(j) \geq 7$, and r is odd, it is true that $r \geq 17$. Moreover, we have that r divides one of the numbers $p, q + 1, q - 1$ and therefore $q + 1 \geq 17$. Since q is odd, we obtain that $q \geq 17$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L . It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq 2^{2t+2d}$ by Lemma 3.10 (iii). Since $q \geq 17$, we have that $17^t/2 \leq b \leq 2^{2t+2d}$, and hence $2^{4t} < 2^{2t+2d+1}$. So $2t + 2d \geq 4t$, and hence $d \geq t$. As above, we obtain that $r_j(u) \geq 17$ for any $j \in J$. Observe that at most one $r_j(u)$ is equal to p , and hence either $q + 1$ or $q - 1$ is divisible by at least $(d - 1)/2$ different primes $r_j(u)$. So $q + 1 \geq 17^{(d-1)/2} > 2^{2(d-1)} + 1$, and hence $q > 2^{2(d-1)}$. Therefore $b \geq q^t/2 > 2^{2t(d-1)-1}$. Whence we have that $2^{2t(d-1)-1} < 2^{2(t+d)}$. Since $d \geq t$, it is true that $t(d - 1) > t + d$, and hence $2^{2t(d-1)} > 2^{2(t+d)}$; a contradiction. The proposition is proved. \square

Lemma 5.5. *Let $L \in \mathcal{L}$ and q is odd. If $s \in \pi(L)$ is chosen so that $\varphi(s, L) > n/3$, then $t(s, L) = t(s, S)$.*

Proof. This lemma is an analogue of [12, Lemma 7.1], where the assertion is proved for L with $t(L) \geq 23$. The proof is based on Propostions 5, 7, 8 from [12]. To adopt this proof for $L \in \mathcal{L}$, we can use Propostion 4.3, Propostion 5.2, and Lemma 4.5 respectively. \square

Following [12], define

$$M(L) = \{i \mid \varphi(r_i(q), L) > n/3, t(r_i(q), L) < t(L)\},$$

$$M(S) = \{i \mid \varphi(r_i(u), S) > m/3, t(r_i(u), S) < t(S)\}.$$

Lemma 5.6. *Let $L = L_n^\varepsilon(q)$, where $n \geq 27$ and $i \in M(L)$. Then*

- (i) $t(r_i(\varepsilon q), L) = i$;
- (ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Let $i \in M(L)$. Arguing as in the proof of [12, Lemma 7.3], we obtain the equality $t(r_i(\varepsilon q), L) = i$. If $j \leq n/3$ then the similar reasoning shows that $t(r_j(\varepsilon q), L) \leq j$. Now we have that $t(r_j(\varepsilon q), L) \leq j \leq n/3 < i = t(r_i(\varepsilon q), L)$, and hence (ii) is proved. \square

Lemma 5.7. *Let $L \in \{S_{2n}(q), O_{2n+1}(q)\}$, where $n \geq 17$ and $i \in M(L)$. Then*

$$(i) \ t(r_i(q), L) = \begin{cases} \lfloor \frac{3\eta(i)+2}{2} \rfloor & \text{if } n \text{ is even,} \\ \lfloor \frac{3\eta(i)+3}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

In particular, if n is even then $t(r_i(q), L) \not\equiv 0 \pmod{3}$ and if n is odd then $t(r_i(q), L) \not\equiv 2 \pmod{3}$.

- (ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(q), L)$.

Proof. Let $i \in M(L)$. Define a function $\tau(k)$ on positive integer k :

$$\tau(k) = \begin{cases} \lfloor \frac{3\eta(k)+2}{2} \rfloor & \text{if } n \text{ is even,} \\ \lfloor \frac{3\eta(k)+3}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Observe that if k_1, k_2 are integers and $\eta(k_1) > \eta(k_2)$, then $\tau(k_1) > \tau(k_2)$. Arguing as in the proof of [12, Lemma 7.4], we obtain the required equality $t(r_i(q), L) = \tau(i)$. If $j \leq n/3$ then the similar reasoning shows that $t(r_j(\varepsilon q), L) \leq \tau(j)$. Since $\varphi(r_j(q), l) = \eta(j) \leq n/3 < \varphi(r_i(q), L) = \eta(i)$, we have that $\tau(j) < \tau(i)$, and hence $t(r_i(q), L) < t(r_j(q), L)$, so (ii) is proved. \square

Lemma 5.8. Let $L \simeq O_{2n}^+(q)$, where $n \geq 19$ and $i \in M(L)$. Then

$$(i) \ t(r_i(q), L) = \begin{cases} \lfloor \frac{3\eta(i)+1}{2} \rfloor & \text{if } n \text{ is even,} \\ \lfloor \frac{3\eta(i)+2}{2} \rfloor & \text{if } n \text{ is odd and } i \text{ is even,} \\ \lfloor \frac{3\eta(i)+3}{2} \rfloor & \text{if } n \text{ is odd and } i \text{ is odd.} \end{cases}$$

In particular, if n is even then $t(r_i(q), L) \not\equiv 1 \pmod{3}$.

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Similar arguments as in the proof of Lemma 5.7 show that the assertions (i), (ii) follow from the proof of [12, Lemma 7.5]. □

Lemma 5.9. Let $L \simeq O_{2n}^-(q)$, where $n \geq 18$ and $i \in M(L)$. Then

$$(i) \ t(r_i(q), L) = \begin{cases} \lfloor \frac{3\eta(i)+4}{2} \rfloor & \text{if } n \text{ is even,} \\ \lfloor \frac{3\eta(i)+2}{2} \rfloor & \text{if } n \text{ is odd and } i \text{ is odd,} \\ \lfloor \frac{3\eta(i)+3}{2} \rfloor & \text{if } n \text{ is odd and } i \text{ is even.} \end{cases}$$

In particular, if n is even then $t(r_i(q), L) \not\equiv 1 \pmod{3}$.

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Similar arguments as in the proof of Lemma 5.7 show that the assertions (i), (ii) follow from the proof of [12, Lemma 7.6]. □

Put

$$T(L) = \{t(r_i(q), L) \mid i \in M(L)\}.$$

Recall that L is a group from the assumption of Theorem 1.3 if and only if $L \in \mathcal{L}$ and q is odd, since this is a definition of \mathcal{L} . Note that the equality $t(L) = t(S)$ implies that the nonabelian composition factor S of G also lies in \mathcal{L} . Define the following classes for groups from \mathcal{L} .

$$\begin{aligned} \mathcal{Y}_0 &= \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \subseteq T(L)\}; \\ \mathcal{Y}_1 &= \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-2, t-3\}\}; \\ \mathcal{Y}_2 &= \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-1, t-3\}\}; \\ \mathcal{Y}_3 &= \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-1, t-2\}\}; \\ \mathcal{X} &= \{O_{36}^-(q)\}; \end{aligned}$$

Lemma 5.10. If $L \in \mathcal{L}$ then either $L \in \mathcal{Y}_i$ with $0 \leq i \leq 3$ or $L \in \mathcal{X}$ according to Table 5. Moreover, if $L \in \mathcal{Y}_3$ then $t-4 \in T(L)$. If $L \in \mathcal{X}$, then $\{t-1, t-2\} \cap T(L) = \{t-2\}$.

Proof. Let $L \simeq O_{2n}^-(q)$. Assume that $n = 4k$, where k is integer and $k \geq 5$. Then by Table 1, we have that $t(L) = 3k + 1$. Lemma 5.9 implies that $t-3 \notin T(L)$. To prove that $L \in \mathcal{Y}_3$ it is sufficient to show that $t-1, t-2 \in T(L)$. Moreover, we need also show that $t-4 \in T(L)$. Using Table 1, we obtain that $r_{2k-1}(q)$, $r_{4k-4}(q)$, and $r_{2k-3}(q)$ are small with respect to L . Since $k \geq 5$, we have that $2k-3 = \eta(2k-3) > 4k/3 = n/3$, so $2k-1, 4k-4, 2k-3 \in M(L)$. By Lemma 5.9, we have that $t(r_{2k-1}(q), L) = 3k = t-1$, $t(r_{4k-4}(q), L) = 3k-1 = t-2$, and $t(r_{2k-3}(q), L) = 3k-3 = t-4$, as required. Assume that $n = 4k+1$, then $k \geq 5$ and $t(L) = 3k+1$ by Table 1. To prove that $L \in \mathcal{Y}_0$, we need to show that $t-1, t-2$, and $t-3$ are in $T(L)$. Using Table 1, we obtain that $4k-2, 2k-1$, and $4k-4$ are small with respect to L . Since $k \geq 5$, we have

that $\eta(4k - 4) > (4k + 1)/3 = n/3$, and hence $4k - 2, 2k - 1, 4k - 4 \in M(L)$. Now Lemma 5.9 implies that $t(r_{4k-2}(q), L) = 3k = t - 1$, $t(r_{2k-1}(q), L) = 3k - 1 = t - 2$, and $t(r_{4k-4}(q), L) = 3k - 2 = t - 3$, as required. Assume that $n = 4k + 2$. Then $k \geq 4$ and $t(L) = 3k + 2$ by Table 1. Note that $k = 4$ corresponds to $L \simeq O_{36}^-(q)$. Lemma 5.9 implies that $t - 1 \notin M(L)$. So to prove that $L \in \mathcal{Y}_1$ we need to show that $t - 2, t - 3 \in T(L)$. Note that $r_{2k-1}(q)$ and $r_{4k-4}(q)$ are small with respect to L by Table 1. If $k > 4$, we have that $2k - 2 = \eta(4k - 4) > (4k + 2)/3 = n/3$, so $4k - 4, 2k - 1 \in M(L)$. In the case $k = 4$ we have that $2k - 1 = \eta(2k - 1) > (4k + 2)/3 = n/3$, so $2k - 1 \in M(L)$. Lemma 5.9 implies that $t(r_{2k-1}(q), L) = 3k = t - 2$, $t(r_{4k-4}(q), L) = 3k - 1 = t - 3$, as required. Assume that $n = 4k + 3$. By Table 1, we have that $k \geq 4$, $t(L) = 3k + 3$. To prove that $L \in \mathcal{Y}_0$, we need to show that $t - 1, t - 2, t - 3 \in M(L)$. Using Table 1, we obtain that $r_{2k+1}(q), r_{4k}(q)$, and $r_{4k-2}(q)$ are small with respect to L and since $\eta(4k - 2) = 2k - 1 > (4k + 3)/3 = n/3$, we obtain that $4k, 4k - 2, 2k + 1 \in M(L)$. Lemma 5.9 implies that $t(r_{2k+1}(q), L) = 3k + 2 = t - 1$, $t(r_{4k}(q), L) = 3k + 1 = t - 2$, and $t(r_{4k-2}(q), L) = 3k = t - 3$, as required.

All other cases for L are considered similarly. If $L \in \mathcal{Y}_i$ in Table 5 then we have that $t - i \notin T(L)$ by Lemmas 5.7, 5.8, 5.9. The last column of Table 5 contains elements of $\{t - 1, t - 2, t - 3\} \cap T(L)$ if L is not in \mathcal{Y}_3 and $t - 1, t - 2, t - 4$ otherwise. \square

TABLE 5. Description of classes \mathcal{X} and \mathcal{Y}_i with $0 \leq i \leq 3$

| L | class | $t(L)$ | elements of $T(L)$ |
|--|-----------------|----------|---|
| $L_n^\varepsilon(q), n = 2k, k \geq 14$ | \mathcal{Y}_0 | k | $t(r_{k-1}, L) = k - 1, t(r_{k-2}, L) = k - 2, t(r_{k-3}, L) = k - 3$ |
| $L_n^\varepsilon(q), n = 2k + 1, k \geq 13$ | \mathcal{Y}_0 | $k + 1$ | $t(r_k, L) = k, t(r_{k-1}, L) = k - 1, t(r_{k-2}, L) = k - 2$ |
| $S_{2n}(q), O_{2n+1}(q), n = 4k, k \geq 5$ | \mathcal{Y}_1 | $3k + 1$ | $t(r_{2k-1}, L) = 3k - 1, t(r_{4k-4}, L) = 3k - 2$ |
| $S_{2n}(q), O_{2n+1}(q), n = 4k + 1, k \geq 4$ | \mathcal{Y}_3 | $3k + 2$ | $t(r_{4k}, L) = 3k + 1, t(r_{2k-1}, L) = 3k, t(r_{4k-4}, L) = 3k - 2$ |
| $S_{2n}(q), O_{2n+1}(q), n = 4k + 2, k \geq 4$ | \mathcal{Y}_2 | $3k + 2$ | $t(r_{4k}, L) = 3k + 1, t(r_{2k-1}, L) = 3k - 1$ |
| $S_{2n}(q), O_{2n+1}(q), n = 4k + 3, k \geq 4$ | \mathcal{Y}_1 | $3k + 3$ | $t(r_{4k}, L) = 3k + 1, t(r_{2k-1}, L) = 3k$ |
| $O_{2n}^+(q), n = 4k, k \geq 5$ | \mathcal{Y}_2 | $3k$ | $t(r_{2k-1}, L) = 3k - 1, t(r_{4k-4}, L) = 3k - 3$ |
| $O_{2n}^+(q), n = 4k + 1, k \geq 5$ | \mathcal{Y}_0 | $3k + 1$ | $t(r_{2k-1}, L) = 3k, t(r_{4k-2}, L) = 3k - 1, t(r_{4k-4}, L) = 3k - 2$ |
| $O_{2n}^+(q), n = 4k + 2, k \geq 5$ | \mathcal{Y}_3 | $3k + 1$ | $t(r_{4k}, L) = 3k, t(r_{2k-1}, L) = 3k - 1, t(r_{4k-4}, L) = 3k - 3$ |
| $O_{2n}^+(q), n = 4k + 3, k \geq 4$ | \mathcal{Y}_0 | $3k + 3$ | $t(r_{4k+2}, L) = 3k + 2, t(r_{4k}, L) = 3k + 1, t(r_{2k-1}, L) = 3k$ |
| $O_{2n}^-(q), n = 4k, k \geq 5$ | \mathcal{Y}_3 | $3k + 1$ | $t(r_{2k-1}, L) = 3k, t(r_{4k-4}, L) = 3k - 1, t(r_{2k-3}, L) = 3k - 3$ |
| $O_{2n}^-(q), n = 4k + 1, k \geq 5$ | \mathcal{Y}_0 | $3k + 1$ | $t(r_{4k-2}, L) = 3k, t(r_{2k-1}, L) = 3k - 1, t(r_{4k-4}, L) = 3k - 2$ |
| $O_{2n}^-(q), n = 4k + 2, k \geq 5$ | \mathcal{Y}_1 | $3k + 2$ | $t(r_{2k-1}, L) = 3k, t(r_{4k-4}, L) = 3k - 1$ |
| $O_{36}^-(q)$ | \mathcal{X} | 14 | $t(r_7, L) = 12$ |
| $O_{2n}^-(q), n = 4k + 3, k \geq 4$ | \mathcal{Y}_0 | $3k + 3$ | $t(r_{2k+1}, L) = 3k + 2, t(r_{4k}, L) = 3k + 1, t(r_{4k-2}, L) = 3k$ |

As a consequence we obtain that $\mathcal{L} = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{X}$.

Lemma 5.11. *If $S \in \mathcal{Y}_i$ with $1 \leq i \leq 3$, then $L \in \mathcal{Y}_i \cup \mathcal{X}$. If $L \in \mathcal{X}$, then $S \notin \mathcal{Y}_2$. In particular, if $L \in \mathcal{Y}_0$, then $S \in \mathcal{Y}_0 \cup \mathcal{X}$.*

Proof. Assume that $S \in \mathcal{Y}_i$, where $i > 0$. Then $t - i \notin T(S)$. Let $L \notin \mathcal{Y}_i$ and $L \not\cong O_{36}^-(q)$. Then $t - i \in T(L)$ by Lemma 5.10. So there exists $r \in \pi(L)$ such that $e(r, q) \in M(L)$ and $t(r, L) = t - i$. Lemma 5.5 implies that $t(r, S) = t - i$. Since $S \in \mathcal{Y}_i$, we have that $e(r, S) \leq m/3$. By Lemmas 5.7, 5.8, 5.9, we have that

$t(r, S) < t(s, S)$ for any $t(s, S) \in T(S)$. Lemma 5.10 implies that the minimal element in $T(S)$ is less than $t - i$. So $t(r, S) < t - i$; a contradiction.

Let $L \in \mathcal{X}$, so $L \simeq O_{36}^-(q)$. Assume that $S \in \mathcal{Y}_2$ and so $t - 2 \notin T(S)$. By Lemma 5.10, there exists $r \in \pi(L)$ such that $e(r, q) \in M(L)$ and $t(r, L) = t - 2$. Lemma 5.5 implies that $t(r, S) = t - 2$. Therefore we have that $e(r, S) \leq m/3$. As above we have that $t(r, S) < t(s, S)$, where $t(s, S) \in T(S)$. So $t(r, S) < t - 2$; a contradiction. \square

Lemma 5.12. *If $L \in \mathcal{L}$ and $L \simeq L_n^\varepsilon(q)$, where q is odd, then $S \simeq L$.*

Proof. Assume the contrary and let at first $S \simeq L_m^\tau(u)$, where $\tau \in \{+, -\}$ and $(n, \varepsilon, q) \neq (m, \tau, u)$. Proposition 5.2 yields the equality $t(L) = t(S)$ and so by Table 1, we have that $m \in \{n-1, n, n+1\}$. Suppose that $m = n-1$. If n is odd then $t(S) \neq t(L)$ by Table 1. Therefore n is even. Using Table 1, we have that $|J(L)| = t(L) + 1$ and $|J(S)| = t(S) = t(L)$. By Lemma 5.5, we obtain that $t(r_n(\varepsilon q), S) = t(r_{n/2}(\varepsilon q), S) = t(S)$. Therefore primes $r_n(\varepsilon q)$ and $r_{n/2}(\varepsilon q)$ are large with respect to S and adjacent in $GK(S)$. Since $|J(S)| = t(S)$, we have that $e(r_n(\varepsilon q), \tau u) = e(r_{n/2}(\varepsilon q), \tau u)$. Observe that $r_{n/2-1}(\varepsilon q)$ is adjacent to $r_{n/2}(\varepsilon q)$ and nonadjacent to $r_n(\varepsilon q)$ in $GK(L)$. By Lemmas 5.5, 5.6, we have that $r_{n/2-1}(\varepsilon q) \in R_k(\tau u)$ for some k and let $r_{n/2}(\varepsilon q) \in R_l(\tau u)$. Since $r_{n/2-1}(\varepsilon q), r_{n/2}(\varepsilon q), r_n(\varepsilon q)$ are coprime to $|\overline{G}/S| \cdot |K|$, we obtain that $r_{n/2-1}(\varepsilon q)$ is adjacent to $r_{n/2}(\varepsilon q)$ and is nonadjacent to $r_n(\varepsilon q)$ in $GK(S)$. However, existence of an edge between representatives from $R_k(\tau u)$ and $R_l(\tau u)$ depends only on k and l but not on representatives; a contradiction. Thus we have that $m \geq n$.

Put $i = 12$ if $n \leq 32$ and $i = 16$ otherwise. Note that $n/3 < i < n/2$, and hence $t(r_i(\varepsilon q), L) < t(L)$. By Lemmas 5.5, 5.6, we obtain that $i = t(r_i(\varepsilon q), L) = t(r_i(\varepsilon q), S)$, and hence $R_i(\varepsilon q) \subseteq R_i(\tau u)$. So $k_i(\varepsilon q)$ divides $k_i(\tau u)$. If $i = 16$ then Lemma 2.5 yields that $8q^8 \leq u^8$. Let $i = 12$. Since $q \neq u$, we have that $k_{12}(q) \neq k_{12}(u)$. So $k_i(\varepsilon q)$ is a proper divisor of $k_i(\tau u)$. By Fermat's little theorem, we obtain that $13(q^4 - q^2 + 1) \leq u^4 - u^2 + 1$. Since $q^4 - q^2 + 1 > q^4/2$, we have that $6q^4 < u^4$, and hence $36q^8 < u^8$. Thus in the all cases it is true that $8q^8 < u^8$. Lemma 3.9 yields that some element x from $m(S)$ divides some element y from $m(L)$. By Lemma 3.8, we obtain that $(u^{m-1} - 1)/(2m) \leq x \leq y \leq 2q^{n-1}$. So $u^{m-1} - 1 \leq 4mq^{n-1}$. Since $n \leq m \leq n + 1$ and $n \leq 44$, we have that $u^{n-1} \leq 4(n + 1)q^{n-1} + 1 < 181q^{n-1}$. On the other hand, we have that $8q^8 < u^8$, and hence $8^{(n-1)/8}q^{n-1} < u^{n-1} < 181q^{n-1}$; a contradiction, since $8^{(n-1)/8} > 8^3 > 181$.

Assume that S is an orthogonal or symplectic group. Lemma 5.11 implies that either $S \in \mathcal{X}$ or $S \in \mathcal{Y}_0$. If $S \in \mathcal{X}$ then $T(S) \cap \{t - 1, t - 2\} = \{t - 2\}$ due to Lemma 5.10. Considering an element $r \in \pi(L)$ such that $t(r, L) = t - 1$ we obtain a contradiction similarly as in Lemma 5.11. So $S \in \mathcal{Y}_0$. By Table 5, we have that $S \simeq O_{2m}^\tau(u)$, where m is odd. Now we prove that n is odd. Assume the contrary and n is even. Then $|J(L)| = t(L) + 1$. By Table 1, if $m \equiv 3 \pmod{4}$ then $|J(S)| = t(S)$ and this case is considered similarly as the case $S = L_m^\varepsilon(u)$ and $m = n - 1$ above. If $m \equiv 1 \pmod{4}$ then $|J(S)| = t(S) + 1 = |J(L)|$. The similar arguments with $r_{n/2-1}(\varepsilon q)$ as above show that $e(r_n(\varepsilon q), S) \neq e(r_{n/2}(\varepsilon q), S)$. Since $r_n(\varepsilon q), r_{n/2}(\varepsilon q)$ are large with respect to S and adjacent in $GK(S)$, by Table 1, we obtain that either $R_n(q) \subseteq R_{m-1}(\tau u)$ and $R_{n/2}(q) \subseteq R_{m+1}(\tau u)$ or $R_n(q) \subseteq R_{m+1}(\tau u)$ and $R_{n/2}(q) \subseteq R_{m-1}(\tau u)$. Let $k = e(r_{n/2-1}(\varepsilon q), S)$. Then $r_k(u)$ is small with respect to S . Therefore we obtain that $\eta(k) \leq (m - 3)/2$ by Table 1. Lemma 3.2 implies that $r_k(u)$ is adjacent to both $r_{m-1}(\tau u)$ and $r_{m+1}(\tau u)$; a contradiction, since $r_{n/2-1}(\varepsilon q)$ and $r_n(\varepsilon q)$ are nonadjacent in $GK(L)$. Thus n is odd. Observe that if n is a prime then by Lemma 3.2, a prime $r_n(\varepsilon q)$ is nonadjacent to $r_i(\varepsilon q)$, where $i \neq n$. So $GK(L)$ is disconnected and $R_n(\varepsilon q)$ is a connected component. Therefore $GK(S)$ is also disconnected. The classification of nonabelian finite simple groups with disconnected prime graph was obtained in [7], [22]. The full list of nonabelian simple groups with a disconnected prime graph can be found, e.g., in [8, Tables 1-3]. In

particular, we obtain that if $GK(O_{2m}^\tau(u))$ with odd m is disconnected, then either m is a prime or $m = 2^t + 1$ for some integer t . Sorting out odd values of n and using that $t(L) = t(S)$, we obtain that only the pairs $(n, m) = (29, 19)$, $(35, 23)$, and $(43, 29)$ are possible.

Let $(n, m) = (35, 23)$. By Lemmas 5.6, 5.8, 5.9, we have that $t(r_{17}(\varepsilon q), L) = 17 = R_{11}(-\tau u)$, and hence $R_{17}(\varepsilon q) \subseteq R_{11}(-\tau u)$. Observe that $r_{17}(\varepsilon q)$ and $r_{34}(\varepsilon q)$ are adjacent in $GK(L)$. Since $r_{34}(\varepsilon q)$ is large with respect to L and $|J(O_{46}^\tau(u))| = t(O_{46}^\tau(u))$, we have that $t(r_{34}(\varepsilon q), S) = t(S)$ and $R_{34}(\varepsilon q) \subseteq R_i(u)$ for some i such that $\eta(i) \geq 11$ due to Table 1. Then $r_i(u)$ is adjacent with $r_{11}(-\tau u)$. By the adjacent criterion, we obtain that $22 + 2\eta(i) \leq 46$, and hence $\eta(i) \leq 12$. Observe that $r_{18}(\varepsilon q)$ is large with respect to L , adjacent with $r_{17}(\varepsilon q)$ in $GK(L)$, and nonadjacent with $r_{34}(\varepsilon q)$. So $r_{34}(\varepsilon q) \notin R_{11}(-\tau u)$. Since $\eta(i) \geq 11$, we have that either $r_{34}(\varepsilon q) \in R_{11}(\tau u)$ or $r_{34}(\varepsilon q) \in R_{24}(u)$. Note that $r_{11}(\tau u)$ and $r_{24}(u)$ are adjacent with $r_9(u)$ in $GK(S)$, so $r_{34}(\varepsilon q)$ is adjacent with $r_9(u)$ in $GK(L)$. Observe that $9 \in M(S)$, therefore by Lemmas 5.8, 5.9, we have that $t(r_9(u), S) \in \{14, 15\}$. Since $14, 15 \in M(L)$ and $t(r_{14}(\varepsilon q)) = 14$, $t(r_{15}(\varepsilon q)) = 15$, we obtain that either $R_{14}(\varepsilon q) \subseteq R_9(u)$ or $R_{15}(\varepsilon q) \subseteq R_9(u)$. So we can choose $r_9(u) \in R_{14}(\varepsilon q) \cup R_{15}(\varepsilon q)$. We get a contradiction, since $r_{34}(\varepsilon q)$ is adjacent to neither $r_{14}(\varepsilon q)$ nor $r_{15}(\varepsilon q)$ in $GK(L)$.

Assume that $(n, m) = (29, 19)$ or $(43, 29)$. Recall that $GK(L), GK(S)$ are disconnected. Moreover, it is showed in [8, Lemma 4] that $R_n(q)$ and $R_m(\tau u)$ are connected components that do not contain 2 in $GK(L)$ and $GK(S)$ respectively. So $k_n(\varepsilon q) = k_m(\tau u)$. Let $\tau = +$. Then by [8, Table 1], we have that $k_n(\varepsilon q) = \frac{q^n - \varepsilon 1}{(q - \varepsilon)(q - \varepsilon 1, n)}$, $k_m(u) = \frac{u^m - 1}{u - 1}$, and $u \in \{2, 3, 5\}$. Let r be a prime divisor of $k_m(u)$. Then $r \in R_n(\varepsilon q)$, and hence $r \equiv 1 \pmod{n}$. So $k_m(u) \equiv 1 \pmod{n}$, and hence $u^m - u$ is divisible by n . It is not hard to check that n does not divide $u^m - u$ under our assumptions. Let $\tau = -$. Then by [8, Table 1], we obtain that $k_m(-u) = (3^m + 1)/4$. By the similar arguments as in the previous case, we have that $k_m(-u) - 1$ is divisible by n , what is not true under our assumptions; a contradiction. \square

Lemma 5.13. *Let $L \in \mathcal{L}$ and q is odd.*

- (i) *If $L \simeq O_{2n}^\varepsilon(q)$, $S \simeq O_{2m}^\tau(u)$ then $n - m$ is even.*
- (ii) *If $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ then $S \not\subseteq O_{2m}^\varepsilon(u)$.*
- (iii) *If $L \simeq O_{2n}^\varepsilon(q)$ then $S \notin \{O_{2m+1}(u), S_{2m}(u)\}$.*

Proof. If $L \simeq O_{2n}^\varepsilon(q)$, where n is odd, then $L \in \mathcal{Y}_0$. Moreover, by Table 1, we have that $t(L) \geq 15$, so $S \notin \mathcal{X}$. By Lemma 5.11, we obtain that $S \in \mathcal{Y}_0$. Table 5 yields that $S \not\subseteq O_{2m}^\tau(u)$, where m is even, and $S \notin \{O_{2n+1}(u), S_{2n}(u)\}$. Let $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ and $S \simeq O_{2m}^\varepsilon(u)$, where m is even. Since $t(L) = t(S)$, we get a contradiction with Table 5 and Lemma 5.11. By the same reasons, we get a contradiction if $L \simeq O_{2n}^\varepsilon(u)$, where n is even, $(n, \varepsilon) \neq (18, -)$, and $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Assume that $L = O_{36}^-(q)$ and $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Then Lemma 5.11 yields that $S \notin \mathcal{Y}_2$. Since $t(S) = 14$, we have that $r_9(q), r_{16}(q), r_{18}(q)$ are large with respect to L and pairwise adjacent in $GK(L)$. Lemma 5.5 yields that $r_9(q), r_{16}(q), r_{18}(q)$ are large with respect to S . By Table 1, we have that $J(S) = E(S)$. So there exists i such that $r_9(q), r_{16}(q), r_{18}(q) \in R_i(u)$. Therefore $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q)$ divides $k_i(u)$, so $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q) \in \omega(L)$. The spectrum description of $O_{36}^-(q)$ with odd q contains in [2, Corollaries 8-9], in particular, we have that $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q) \notin \omega(O_{36}^-(q))$; a contradiction. Thus (iii) follows.

Assume that $S \simeq O_{2m}^\tau(u)$, where m is odd, $\tau \in \{+, -\}$, and either $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ or n is even and $L \simeq O_{2n}^\pm(q)$. By Lemma 3.9, we have that there exist $x \in m(S)$ and $y \in m(L)$ such that x divides y . Moreover, Lemma 3.8 yields that $u^m/8 \leq x$ and since q is odd, $y \leq q^n$. Proposition 5.2 implies that $t(S) = t(L)$ and using Table 1 we obtain that $n - 1 \leq m \leq n + 1$. Whence we have the inequality $u^{n-1} \leq 8q^n$.

Now we prove that $q^2 < 2u$. Since $L \in \mathcal{L}$, we have that $n \leq 30$. Assume that $n \leq 20$. Then by Table 1, we have that $7, 14 \in M(L)$. Using Lemmas 5.7, 5.8, 5.9, we obtain that $t(r_7(q), L) = t(r_{14}(q), L)$ and either $t(r_7(q), L) = 11$ or $t(r_7(q), L) = 12$. Lemma 5.5 implies that $t(r_7(q), L) = t(r_7(q), S)$ and $t(r_{14}(q), L) = t(r_{14}(q), S)$. Observe that by Lemmas 5.8, 5.9, if s is a prime and $t(s, S) = 11$ then $e(s, u) = 14$ in the case $\tau = +$, and $e(s, u) = 7$, when $\tau = -$. Similarly, we have that $t(s, S) = 12$ is equivalently to $e(s, u) = 7$, when $\tau = +$, and $e(s, u) = 14$, when $\tau = -$. So either $R_7(q), R_{14}(q) \subseteq R_7(u)$ or $R_7(q), R_{14}(q) \subseteq R_{14}(u)$. Therefore $k_7(q)k_{14}(q)$ divides either $k_7(u)$ or $k_{14}(u)$. Observe that $k_7(q)k_{14}(q) = \frac{q^{14}-1}{(q^2-1)(q-1,7)(q+1,7)} \geq (q^{12}+q^{10}+\dots+q^2+1)/7 \geq q^{12}/7$. It is clear that both numbers $k_7(u)$ and $k_{14}(u)$ are not greater than $u^6+u^5+u^4+u^3+u^2+u+1$. So $q^{12}/7 < 2u^6$, and hence $q^{12} < 14u^6$. Since $2^6 > 14$, we obtain the required inequality $q^2 < 2u$. Assume that $20 < n \leq 26$. Then by Table 1, we have that $9, 18 \in M(L)$. Arguing as above, we obtain that $k_9(q)k_{18}(q)$ divides either $k_9(u)$ or $k_{18}(u)$. It is clear that $k_9(u) \leq u^6+u^3+1 < 2u^6$ and $k_{18}(u) \leq u^6-u^3+1 < 2u^6$. Moreover, $k_9(q)k_{18}(q) = \frac{(q^6+q^3+1)(q^6-q^3+1)}{(q-1,3)(q+1,3)} \geq (q^{12}+q^6+1)/3 \geq q^{12}/3$. So we have that $q^{12} < 6u^6$, and hence $q^2 < 2u$, as claimed. Let finally $27 \leq n \leq 30$. Then by Table 1, we have that $11, 22 \in M(L)$. Arguing as above, we obtain that $k_{11}(q)k_{22}(q)$ divides either $k_{11}(u)$ or $k_{22}(u)$. It is clear that $k_{11}(u) < 2u^{10}$ and $k_{22}(u) < 2u^{10}$. Moreover, $k_{11}(q)k_{22}(q) = \frac{q^{22}-1}{(q^2-1)(q-1,11)(q+1,11)} \geq q^{20}/11$. So we have that $q^{20} < 22u^{10}$, and hence $q^2 < 2u$, as claimed. Now we prove that obtained inequalities $u^{n-1} \leq 8q^n$ and $q^2 < 2u$ contradict to each other. Observe that $q^2 < 2u$ is equivalent to $q^{2n-2} < 2^{n-1}u^{n-1}$. Therefore we have that $q^{2n-2} < 2^{n+2}q^n$, and hence $q^{n-2} < 2^{n+2}$. Since $q \geq 3$, we have $q^{n-2} \geq 3^{n-9}3^7 > 2^{n-9}2^{11} > 2^{n+2}$; a contradiction. The lemma is proved. \square

Lemma 5.14. *Let $L \in \mathcal{L}$ and $L \in \{O_{2n+1}(q), S_{2n}(q), O_{2n}^\pm(q)\}$ with odd q . Then $S \not\cong L_m^\tau(u)$, where $\tau \in \{+, -\}$.*

Proof. Assume the contrary and let $S \cong L_m^\tau(u)$, where $\tau \in \{+, -\}$. Since $t(S) \leq 22$, we have that $m \leq 44$. Let at first $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ or n is even and $L \cong O_{2n}^\pm(q)$. Lemma 3.9 implies that for some $x \in m(S)$, $y \in m(L)$ it is true that x divides y . By Lemma 3.8, we have that $x \geq (u^{m-1}-1)/(2m)$ and $y \leq 2q^n$. Therefore $u^{m-1} \leq 4mq^n + 1 \leq 180q^n$.

Now we prove that $q < u$. Observe that $17 \leq n \leq 30$ and by Table 1, we have that $\{7, 9, 11\} \cap M(L) \neq \emptyset$. Let i be the minimal element from this intersection and note that $2i \in M(L)$. Let $j = t(r_i(q), L)$. Then Lemmas 5.7, 5.8, 5.9 imply that $j \in \{11, 12\}$, $j \in \{14, 15\}$, or $j \in \{17, 18\}$, if $i = 7, 9$ or 11 respectively. Lemma 5.6 yields that $t(r, S) = j$ is equivalent to $r \in R_j(\tau u)$. So $R_i(q), R_{2i}(q) \subseteq R_j(\tau u)$, and hence $k_i(q)k_{2i}(q)$ divides $k_j(\tau u)$. If $i = 7$ and $d = (q-1, 7)(q+1, 7)$, then $k_7(q)k_{14}(q) = \frac{q^{14}-1}{d(q^2-1)} \geq q^{12}/d$ and $k_j(\tau u) \leq 2u^{10}$. So $q^{12} \leq 2du^{10}$. If $d = 1$ then since $q^2 > 2$, we obtain that $q < u$. If $d > 1$ then $d = 7$, and hence $q > 7$. In this case $q^2 > 2d$, and hence again $q < u$. If $i = 9$ then $k_9(q)k_{18}(q) \geq (q^{12}+q^6+1)/3 \geq q^{12}/3$ and $k_j(\tau u) \leq u^8$. So $q^{12} \leq 3u^6$, and hence $q < u$. If $i = 11$ then then $k_{11}(q)k_{22}(q) \geq \frac{q^{22}-1}{11(q^2-1)} \geq q^{20}/11$ and $k_j(\tau u) \leq 2u^{16}$. So $q^{20} \leq 22u^{16}$, and hence $q < u$. Thus we have the inequality $q < u$ in all cases. We proved above that $u^{m-1} < 180q^n$. So $q^{m-1} < 180q^n$, and hence $q^{m-n-1} < 180$. Since $t(S) = t(L)$, it is not hard to check using Table 1 that $m-n > 5$, in particular $q^{m-n-1} \geq q^5 \geq 3^5 = 243$; a contradiction.

Consider now the case $L \cong O_{2n}^\varepsilon(q)$, where n is odd and $\varepsilon \in \{+, -\}$. First we prove that $q \geq 7$. Assume that $q < 7$. Then since q is odd, we have that either $q = 3$ or $q = 5$. If $n < 24$ then by Table 1, we have that $16 \in M(L)$. Lemma 5.5 implies that $t(r_{16}(q), L) = t(r_{16}(q), S)$. By Lemmas 5.8, 5.9, we obtain that $t(r_{16}(q), L) = 13$ and so $r_{16}(q) \in R_{13}(\tau u)$ due to Lemma 5.6. Observe that $17 \in R_{16}(3)$ and $17 \in R_{16}(5)$. So $17 \in R_{13}(\tau u)$; a contradiction with Lemma 3.3. If $n \geq 24$ then by Table 1 we have that $11, 22 \in M(L)$. Lemma 5.5 implies that $t(r_{11}(q), L) = t(r_{11}(q), S)$ and $t(r_{22}(q), L) = t(r_{22}(q), S)$. By Lemmas 5.8, 5.9, we obtain that

$t(r_{11}(q), L), t(r_{22}(q), L) \in \{17, 18\}$ and so $r_{11}(q), r_{22}(q) \in R_{17}(\tau u) \cup R_{18}(\tau u)$ due to Lemma 5.6. Observe that $23 \in R_{11}(3)$ and $23 \in R_{22}(5)$. So either $23 \in R_{17}(\tau u)$ or $23 \in R_{18}(\tau u)$; a contradiction with Lemma 3.3. Thus we have that $q \geq 7$.

Now Lemma 3.8 yields that there exist $x \in m(S), y \in m(L)$ such that x divides y . Since q is odd, by Lemma 3.8 we obtain that $y \leq q^n$. Moreover, we have that $\frac{u^{m-1}-\tau 1}{2(m, u-\tau 1)} \leq x$. Observe that $(u+1)u^{m-3} < u^{m-1} - 1$, and hence $u^{m-3}/2 < \frac{u^{m-1}-\tau 1}{2(m, u-\tau 1)}$. So $u^{m-3} < 2q^n$. Now we prove that $q^6 < 7u^6$. Let at first $n < 27$. Then by Table 1, we have that $18 \in M(L)$ if $\varepsilon = +$ and $9 \in M(L)$ if $\varepsilon = -$. Lemmas 5.8, 5.9 yield that $t(r_9(-\varepsilon q), L) = 14$. So $R_9(-\varepsilon q) \subseteq R_{14}(\tau u)$ due to Lemma 5.6, and hence $k_9(-\varepsilon q)$ divides $k_{14}(\tau u)$. Observe that $k_9(-\varepsilon q) = (q^6 - \varepsilon q^3 + 1)/(q + \varepsilon 1, 3)$ and $k_{14}(\tau u)$ divides $u^6 - \tau u^5 + u^4 - \tau u^3 + u^2 - \tau u + 1$. If $q \leq u$ then the inequality $q^6 < 7u^6$ obviously holds. Let $q \geq u + 1$. Then $q^6 - \varepsilon q^3 + 1 > (u+1)^6 - \varepsilon(u+1)^3 + 1 > u^6 - \tau u^5 + u^4 - \tau u^3 + u^2 - \tau u + 1$. So $k_{14}(\tau u)/k_9(-\varepsilon q) < 3$. Since $k_9(-\varepsilon q)$ is odd, we obtain that $q^6 - \varepsilon q^3 + 1 = 3(u^6 - \tau u^5 + u^4 - \tau u^3 + u^2 - \tau u + 1)$. Since $q \geq 7$, we have that $q^6 - \varepsilon q^3 + 1 > 6/7q^6$. On the other hand, $u^6 - \tau u^5 + u^4 - \tau u^3 + u^2 - \tau u + 1 < 2u^6$. So $6/7q^6 < 6u^6$, and hence $q^6 < 7u^6$, as required. Let now $n \geq 27$. Then by Table 1, we have that $11 \in M(L)$ if $\varepsilon = +$ and $22 \in M(L)$ if $\varepsilon = -$. Lemmas 5.8, 5.9 yield that $t(r_{11}(\varepsilon q), L) = 18$. So $R_{11}(\varepsilon q) \subseteq R_{18}(\tau u)$, and hence $k_{11}(\varepsilon q)$ divides $k_{18}(\tau u)$. Observe that $k_{11}(\varepsilon q) \geq \frac{(q^{11}+1)}{11(q+1)} > q^{10}/22$ and $k_{18}(\tau u) < u^6 + u^3 + 1 < 2u^6$. Therefore we have that $q^{10} < 44u^6$, and hence $q^6 < 7u^6$ due to $q \geq 7$. Thus we have that $q^6 < 7u^6$ in all cases.

Now we prove that the obtained inequalities $u^{m-3} \leq 2q^n$ and $q^6 < 7u^6$ contradict to each other. The inequality $q^6 < 7u^6$ is equivalent to $q^{m-3} < 7^{(m-3)/6}u^{m-3}$. So $q^{m-3} < 2 \cdot 7^{(m-3)/6}q^n$ or equivalently $q^{m-n-3} < 2 \cdot 7^{(m-3)/6}$. Therefore $q^{m-n-3} < 7^{(m+3)/6}$. We have that $q \geq 7$ and so to obtain a contradiction it is sufficient to show that $6(m-n-3)/(m+3) > 1$. This inequality is equivalent to $5(m-3) > 6n+6$. By Table 1, we have that $(m+1)/2 \geq t(S)$ and $t(L) \geq (3n+1)/4$. Therefore $(m+1)/2 \geq (3n+1)/4$, and hence $2m+1 \geq 3n$. So $5(m-3) = 2(2m+1) + m - 17 > 6n+6$ due to $m \geq 27$ by Table 1. Thus $5(m-3) > 6n+6$ and as was pointed above this inequality leads to a contradiction. The lemma is proved. \square

Lemma 5.15. *Assume that $L \in \mathcal{L}$ and $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ with odd q . Then $S \in \{O_{2n+1}(q), S_{2n}(q)\}$.*

Proof. By Lemmas 5.13, 5.14, we have that $S \not\cong O_{2m}^\pm(u)$ and $S \not\cong L_m^\pm(u)$. So $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Proposition 4.3, Lemma 5.10, and Table 5 imply that $n = m$.

By Lemma 3.9, we obtain that for some $x \in m(S), y \in m(L)$ it is true that x divides y . Lemma 3.8 implies that $y \leq (q^n + 1)/2$ and $(u^n - 1)/2 \leq x$. Therefore $u^n - 2 \leq q^n$.

Assume that $17 \leq n \leq 23$. Then by Table 1, we obtain that $16 \in M(L)$. Lemmas 5.5, 5.7 yield that $t(r_{16}(q), L) = t(r_{16}(q), S)$ and $t(r_{16}(q), L) = 13$. Therefore $k_{16}(q)$ divides $k_{16}(u)$. Lemma 2.5 implies that $8q^8 < u^8$. We rewrite this as $8^{n/8}q^n \leq u^n$ and since $n \geq 17$, we obtain that $8q^n < u^n$. On the other hand, we have that $u^n \leq q^n + 2$; a contradiction.

Assume that $24 \leq n \leq 26$. Then $9, 18 \in M(L)$. Lemma 5.7 implies that $t(r_9(q), L) = t(r_{18}(q), L)$. Since $t(r_9(q), L) = t(r_9(q), S)$ and $t(r_{18}(q), L) = t(r_{18}(q), S)$, we have that $R_9(q) \cup R_{18}(q) \subseteq R_9(u) \cup R_{18}(u)$. So $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$. Lemma 2.5 implies that $u^{12} > 3q^{12}$. So $u^n > 3^{n/12}q^n \geq 9q^n$. However, we obtained above that $u^n - 2 \leq q^n$; a contradiction.

Assume finally that $n = 27, 28$. Note that $24 \in M(L)$. Lemma 5.7 implies that $t(r_{24}(q), L) = t(r_{24}(q), S)$. So $R_{24}(q) \subseteq R_{24}(u)$ and $k_{24}(q)$ divides $k_{24}(u)$. Lemma 2.5 implies that $u^8 > 12q^8$, and hence $u^n > 12^{n/8}q^n > 12^3q^n$; a contradiction with the inequality $u^n \leq q^n + 2$. The lemma is proved. \square

Lemma 5.16. *Let $L \in \mathcal{L}$ and q is odd. If $L = O_{2n}^+(q)$, where n is even, then $S \in \{O_{2n}^+(q), O_{2n-4}^-(q)\}$ and if $L = O_{2n}^-(q)$, where n is even, then $S \in \{O_{2n}^-(q), O_{2n+4}^+(q)\}$.*

Proof. Let $L = O_{2n}^\varepsilon(q)$, where n is even and $\varepsilon \in \{+, -\}$. Lemmas 5.13, 5.14 yield that $S \simeq O_{2m}^\tau(u)$, where m is even and $\tau \in \{+, -\}$. Since $t(L) = t(S)$, by Lemma 5.10 and Table 5.10, we obtain that if $L = O_{2n}^+(q)$ then $S \in \{O_{2n}^+(u), O_{2n-4}^-(u)\}$ and if $L = O_{2n}^-(q)$ then $S \in \{O_{2n}^-(u), O_{2n+4}^+(u)\}$.

Now we prove that $u = q$. Assume the contrary. Let at first $m = n$, and hence $\tau = \varepsilon$. By Lemma 3.9, we obtain that there exist $x \in m(S)$, $y \in m(L)$ such that x divides y . Lemma 3.8 yields that if $\varepsilon = +$ then $u^{n-1}/4 \leq x$, $y \leq q^{n-1}$, and if $\varepsilon = -$ then $u^n/4 \leq x$, $y \leq q^n$. So either $u^{n-1} < 4q^{n-1}$ or $u^n < 4q^n$. Let $n = 18$. Since $t(O_{36}^+(q)) = 13$ due to Table 1, we conclude that $\varepsilon = -$. Then $7, 14 \in M(L)$. Since $t(r_7(q), L) = t(r_7(q), S)$, we obtain that $R_7(q) \cup R_{14}(q) \subseteq R_7(u) \cup R_{14}(u)$. Using the adjacency criterion from Lemma 3.2, we obtain that if $r_i(q)$ is large with respect to L and adjacent with $r_7(q)$ in $GK(L)$ then $i \in \{9, 11, 16, 18, 20\}$. Similarly, if $r_i(q)$ is large with respect to L and adjacent with $r_{14}(q)$ in $GK(L)$ then $i \in \{9, 16, 18, 20, 22\}$. So we obtain that $R_{11}(q) \cup R_{22}(q) \subseteq R_{11}(u) \cup R_{22}(u)$. Assume that there exist $r, r' \in R_7(q)$ such that $r \in R_7(u)$ and $r' \in R_{14}(u)$. We have that $r_{11}(q) \in R_i(u)$, where $i \in \{11, 22\}$. By the adjacency criterion, we have that $r_{11}(q)$ is adjacent in $GK(S)$ with exactly one of r and r' ; a contradiction, since $r_{11}(q)$ is adjacent with both of them in $GK(L)$. So either $R_7(q) \subseteq R_7(u)$ or $R_7(q) \subseteq R_{14}(u)$. The similar is true for $R_{14}(q)$. Let $\epsilon, \lambda \in \{+, -\}$ such that $(q - \epsilon 1, 7) = 1$ and $R_7(\epsilon q) \subseteq R_7(\lambda u)$. Then $k_7(\epsilon q)$ divides $k_7(\lambda u)$. By Lemma 2.5, we have that $3q^6 < u^6$, and hence $3^{3/2}q^{18} < u^{18}$. We obtained earlier that $u^{18} \leq 4q^{18}$; a contradiction, since $3^{3/2} > 4$. Assume that $20 \leq n \leq 22$. Then $16 \in M(L)$. Lemma 5.5 implies that $t(r_{16}(q), L) = t(r_{16}(q), S)$. So $R_{16}(q) \subseteq R_{16}(u)$, and hence $k_{16}(q)$ divides $k_{16}(u)$. Lemma 2.5 implies that $u^8 > 8q^8$. However, either $u^{n-1} > 8^{(n-1)/8}q^{n-1} > 64q^{n-1}$ or $u^n > 8^{n/8}q^n > 64q^n$; a contradiction with $u^{n-1} \leq 4q^{n-1}$ or $u^n \leq 4q^n$ respectively. If $24 \leq n \leq 26$ then $9, 18 \in M(L)$. Since $t(r_9(q), L) = t(r_{18}(q), L)$, we obtain that $R_9(q) \cup R_{18}(q) \subseteq R_9(u) \cup R_{18}(u)$. Therefore $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$. Lemma 2.5 implies that $3q^{12} < u^{12}$. So either $u^{n-1} > 3^{(n-1)/12}q^{n-1} > 5q^{n-1}$ or $u^n > 3^{n/12}q^n > 5q^n$; a contradiction. Let finally $28 \leq n \leq 30$. Then $24 \in M(L)$. So $t(r_{24}(q), L) = t(r_{24}(q), S)$, and hence $R_{24}(q) \subseteq R_{24}(u)$. Therefore $k_{24}(q)$ divides $k_{24}(u)$ and by Lemma 2.5, we obtain that $12q^{12} < u^{12}$. Thus either $u^{n-1} > 12^{(n-1)/12}q^{n-1} > 144q^{n-1}$ or $u^n > 12^{n/12}q^n > 144q^n$; a contradiction.

Assume that $L = O_{2n}^+(q)$, $S = O_{2n-4}^-(u)$. Since $t(S) = t(L)$, by Table 1 we obtain that $n \equiv 2 \pmod{4}$. By Lemma 3.9, we obtain that for some $x \in m(S)$ and $y \in m(L)$ it is true that x divides y . Lemma 3.8 yields that $u^{n-2}/4 \leq x$ and $y \leq q^{n-1}$. So $u^{n-2} \leq 4q^{n-1}$.

Observe that $n \in \{22, 26, 30\}$. If $n = 22$ then $9, 18 \in M(L)$ and $t(r_9(q), L) = t(r_{18}(q)) = 14$. Since $t(r_{16}(u), S) = 14$, we conclude that $R_9(q) \cup R_{18}(q) \subseteq R_{16}(u)$. Then $k_9(q)k_{18}(q)$ divides $k_{16}(u)$. Since $k_9(q)k_{18}(q) = \frac{(q^6+q^3+1)(q^6-q^3+1)}{(q-1,3)(q+1,3)}$, we obtain that $q^{12}/3 \leq k_9(q)k_{18}(q)$. Since $k_{16}(u) = (u^8 + 1)/(2, u - 1)$, we have that $q^{12} < 3(u^8 + 1) < 4u^8$. So $q^6 \leq 2u^4$. Therefore $q^{30} < 32u^{20} \leq 128q^{21}$; a contradiction with $q \geq 3$. If $n = 26, 30$ then we have that $11, 22 \in M(L)$. Since $t(r_{11}(q), L) = t(r_{22}(q), L) = 17$ and $t(r_{20}(u), S) = 17$, we have that $R_{11}(q) \cup R_{22}(q) \subseteq R_{20}(u)$. Therefore $k_{11}(q)k_{22}(q)$ divides $k_{20}(u)$. Observe that $k_{11}(q)k_{22}(q) = \frac{q^{22}-1}{(q^2-1)(q-1,11)(q+1,11)}$, $k_{20}(u) = (u^8 - u^6 + u^4 - u^2 + 1)/(u^2 + 1, 5)$. So $k_{11}(q)k_{22}(q) \geq (q^{20} + q^{18} + \dots + q^2 + 1)/11 \geq q^{20}/11$ and $k_{20}(u) \leq u^8$. Therefore $q^{20} < 11u^8$. Then $q^{10} < 4u^4$, and hence if $n = 26$ then $q^{60} < 4^6u^{24} < 4^7q^{25}$, and if $n = 30$ then $q^{70} < 4^7u^{28} < 4^8q^{29}$. Again we get a contradiction with $q \geq 3$.

Assume finally that $L = O_{2n}^-(q)$, $S = O_{2n+4}^+(u)$. Since $t(S) = t(L)$, by Table 1 we obtain that $n \equiv 0 \pmod{4}$. By Lemma 3.9, we have that for some $x \in m(S)$, $y \in m(L)$ it is true that x divides y . Lemma 3.8 yields that $u^{n+1}/4 \leq x$ and $y \leq q^n$. Therefore $u^{n+1} \leq 4q^n$. Observe that since $L \in \mathcal{L}$, we have that $n \in \{20, 24, 28\}$. Let $n = 20$ or $n = 24$. Then $9, 18 \in M(L)$. Since $t(r_9(q), L) = t(r_{18}(q), L) = 15$ and $t(r_{20}(u), S) = 15$ we have that $R_9(q) \cup R_{18}(q) \subseteq R_{20}(u)$. Therefore $k_9(q)k_{18}(q)$ divides $k_{20}(u)$. As in the previous case, we obtain that $q^{12}/3 \leq k_9(q)k_{18}(q)$. Since $k_{20}(u) = (u^8 - u^6 + u^4 - u^2 + 1)/(u^2 + 1, 5) < u^8$, we have that $q^{12} < 3u^8 < 4u^8$. So $q^6 < 2u^4$. If $n = 20$ then we conclude that $q^{30} < 2^5u^{20}$. On the other hand, $u^{21} \leq 4q^{20}$, and hence $q^{30} < 128q^{20}$; a contradiction with $q \geq 3$. If $n = 24$ then we have that $q^{36} < 2^6u^{24}$ and $u^{25} < 4q^{24}$. So $q^{36} < 256q^{24}$; a contradiction. Assume that $n = 28$. Then $11, 22 \in M(L)$, $t(r_{11}(q), L) = t(r_{22}(q), L) = 18$, and $t(r_{24}(u), S) = 18$. So $k_{11}(q)k_{22}(q)$ divides $k_{24}(u)$. Above we obtained that $k_{11}(q)k_{22}(q) \geq q^{20}/11$. Since $k_{24}(u) = u^8 - u^4 + 1$, we have that $q^{20} < 11u^8$. Therefore $q^5 < 2u^2$, and hence $q^{70} < 2^{14}u^{28} < 2^{14}u^{29} < 2^{16}q^{28}$; a contradiction with $q \geq 3$. The lemma is proved. \square

Lemma 5.17. *Let $L \in \mathcal{L}$ and $L \in \{O_{2n}^+(q), O_{2n}^-(q)\}$, where n and q are odd. Then $S \in \{O_{2n}^+(q), O_{2n}^-(q)\}$.*

Proof. Let $\varepsilon \in \{+, -\}$ is chosen so that $L = O_{2n}^\varepsilon(q)$. By the hypothesis, we have that $19 \leq n \leq 29$. By Lemmas 5.13, 5.14, we obtain that $S \simeq O_{2m}^\tau(u)$, where m is odd and $\tau \in \{+, -\}$. By Proposition 5.2, we have $t(L) = t(S)$, and hence $m = n$ due to Table 1. Now Lemma 3.9 yields there exist $x \in m(S)$ and $y \in m(L)$ such that x divides y . Moreover, Lemma 3.8 implies that $u^n/8 \leq x$ and $y \leq q^n$. So $u^n \leq 8q^n$. Suppose that $n < 27$. By Table 1, we obtain that $9 \in M(O_{2n}^-(q))$, $18 \in M(O_{2n}^+(q))$. Observe that $t(r_9(-\varepsilon q), L) = 14$ due to Lemmas 5.8, 5.9. By Lemma 5.5, we have that $t(r_9(-\varepsilon q), L) = t(r_9(-\varepsilon q), S)$. So $R_9(-\varepsilon q) \subseteq R_9(-\tau u)$, and hence $k_9(-\varepsilon q)$ divides $k_9(-\tau u)$. Lemma 2.5 implies that $3q^6 < u^6$. So $3^{n/6}q^n < u^n$; a contradiction with the inequality $u^n \leq 8q^n$. Let $n = 27$ or $n = 29$. Then $24 \in M(L)$. By Lemmas 5.8, 5.9, we have that $t(r_{24}(q), L) = 19$. Therefore we obtain that $R_{24}(q) \subseteq R_{24}(u)$, and hence by Lemma 2.5, it is true that $12q^{12} < u^{12}$; a contradiction with the inequality $u^n \leq 8q^n$. The lemma is proved. \square

6. Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2, 1.3. It is not hard to check using the information from Table 1 that if L is a group in odd characteristic from the hypothesis of Theorem A, then $t(L) \geq 23$. Similarly, if L is a classical group from the hypothesis of Theorem 1.2, then either $L \in \{S_{32}(q), O_{33}(q)\}$, where q is a prime power, or $t(L) \geq 14$. It was proved in [15, Theorem 2], that if G is a finite group such that $\omega(G) = \omega(L)$, where $L \in \{S_{32}(q), O_{33}(q)\}$ and q is odd, then G has a unique nonabelian composition factor, and this factor is isomorphic to L . So it remains to prove Theorem 1.2 for a classical group L either in even characteristic, or with $14 \leq t(L) \leq 22$ or equivalently for $L \in \mathcal{L}$.

Let $L \in \mathcal{L}$, p be the characteristic of the underlying field of L , G be a finite group with $\omega(G) = \omega(L)$ and S be a nonabelian composition factor of G . By the main theorem of [16], we may assume that $p \neq 2$. By Lemmas 5.12, 5.15, 5.16, 5.17, we have that if $L \in \mathcal{L}$ and p is odd then S is a classical group of Lie type over a field of characteristic p , and hence Theorem 1.3 is proved. The next two lemmas allow us to finish the proof of Theorem 1.2.

Lemma 6.1. ([6, Theorem 2], [19, Theorem 3]) *Let q be a power of a prime p , L be one of the groups $L_n^\varepsilon(q)$, where $n > 3$ and $(\varepsilon, n, q) \neq (-, 4, 2)$, $S_{2n}(q)$, where $n > 2$ and $(n, q) \notin \{(2, 2), (2, 3)\}$, $O_{2n+1}(q)$, where $n > 3$, or $O_{2n}^\pm(q)$, where $n > 4$, and let G be a finite group with $\omega(G) = \omega(L)$. Suppose that some nonabelian*

composition factor S of G is a group of Lie type over a field of characteristic p . Then either $S \simeq L$ or one of the following holds:

- (i) $L = S_4(q)$, where $q \neq 3^{2k+1}$, and $S = L_2(q^2)$;
- (ii) $L \in \{O_9(q), S_8(q)\}$ and $S = O_8^-(q)$;
- (iii) $\{L, G\} = \{O_8^+(2), S_6(2)\}$;
- (iv) $\{L, G\} = \{O_8^+(3), O_7(3)\}$.

Lemma 6.2. ([5, Corollary 1.1]) *Let L be a finite nonabelian simple group other than $L_4(q)$, $U_3(q)$, $U_4(q)$, $U_5(2)$, and ${}^3D_4(2)$. Let G be a finite group, $1 \neq K \triangleleft G$ and $G/K \simeq L$. Then $\omega(G) \neq \omega(L)$.*

Theorem 1.3 and Lemma 6.1 provide $S \simeq L$, and hence $L \leq G/K \leq \text{Aut } L$, where K is the soluble radical of G . Now Lemma 6.2 yields that $K = 1$, so Theorem 1.2 is proved.

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REFERENCES

- [1] A. A. Buturlakin, Spectra of finite linear and unitary groups, *Algebra Logic*, **47** no. 2 (2008) 91–99.
- [2] A. A. Buturlakin, Spectra of finite symplectic and orthogonal groups, *Siberian Adv. Math.*, **21** no. 3 (2011) 176–210.
- [3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
- [4] P. Erdos, On the coefficients of the cyclotomic polynomial, *Bull. Amer. Math. Soc.*, **52** (1946) 179–184.
- [5] M. A. Grechkoseeva, On element orders in covers of finite simple groups of Lie type, *J. Algebra Appl.*, **14** (2015) 16 pages.
- [6] M. A. Grechkoseeva and A. V. Vasil'ev, On the structure of finite groups isospectral to finite simple groups, *J. Group Theory*, **18** no. 5 (2015) 741–759.
- [7] A. S. Kondrat'ev, On prime graph components of finite simple groups, *Math. USSR-Sb.*, **67** no. 1 (1990) 235–247.
- [8] A. S. Kondrat'ev and V. D. Mazurov, Recognition of alternating groups of prime degree from the orders of their elements, *Siberian Math. J.*, **41** no. 2 (2000) 294–302.
- [9] V. D. Mazurov, Recognition of finite groups by a set of orders of their elements, *Algebra Logic*, **37** no. 6 (1998) 371–379.
- [10] M. Roitman, On Zsigmondy primes, *Proc. Amer. Math. Soc.*, **125** no. 7 (1997) 1913–1919.
- [11] W. J. Shi, The characterization of the sporadic simple groups by their element orders, *Algebra Colloq.*, **1** no. 2 (1994) 156–166.
- [12] A. V. Vasil'ev, On finite groups isospectral to simple classical groups, *J. Algebra*, **423** (2015) 318–374.
- [13] A. V. Vasil'ev, On connection between the structure of a finite group and the properties of its prime graph, *Siberian Math. J.*, **46** no. 3 (2005) 396–404.
- [14] A. V. Vasil'ev and I. B. Gorshkov, On recognition of finite simple groups with connected prime graph, *Siberian Math. J.*, **50** no. 2 (2009) 233–238.
- [15] A. V. Vasil'ev, I. B. Gorshkov, M. A. Grechkoseeva, A. S. Kondrat'ev and A. M. Staroletov, On recognizability by spectrum of finite simple groups of types B_n , C_n , and 2D_n for $n = 2^k$, *Proc. Steklov Inst. Math.*, **267** suppl 1 (2009), 218–233.
- [16] A. V. Vasil'ev and M. A. Grechkoseeva, Recognition by spectrum for simple classical groups in characteristic 2, *Siberian Math. J.*, **56** no. 6 (2015) 1009–1018.
- [17] A. V. Vasil'ev, M. A. Grechkoseeva and V. D. Mazurov, Characterization of the finite simple groups by spectrum and order, *Algebra Logic*, **48** no. 6 (2009) 385–409.

- [18] A. V. Vasil'ev, M. A. Grechkoseeva and V. D. Mazurov, On finite groups isospectral to simple symplectic and orthogonal groups, *Siberian Math. J.*, **50** no. 6 (2009) 965–981.
- [19] A. V. Vasil'ev, M. A. Grechkoseeva and A. M. Staroletov, On finite groups isospectral to simple linear and unitary groups, *Siberian Math. J.*, **52** no. 1 (2011) 30–40.
- [20] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic*, **44** no. 6 (2005) 381–406.
- [21] A. V. Vasil'ev and E. P. Vdovin, Cliques of maximal size in the prime graph of a finite simple group, *Algebra Logic*, **50** no. 4 (2011) 291–322.
- [22] J. S. Williams, Prime graph components of finite groups, *J. Algebra*, **69** (1981) 487–513.
- [23] A. V. Zavarnitsine, Properties of element orders in covers for $L_n(q)$ and $U_n(q)$, *Siberian Math. J.*, **49** no. 2 (2008) 246–256.
- [24] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.*, **3** (1892) 265–284.

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