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Extension of Shift-Invariant Frames for Locally Compact Abelian Groups

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Abstract. Let G be a locally compact abelian group with a uniform lattice subgroup. In this paper, we verify extension of shift-invariant systems in $L^2(G)$ to tight frames. We show that any shift-invariant Bessel sequence with an at most countable number of generators in $L^2(G)$ can be extended to a tight frame for its closed linear span by adding another shift-invariant system with at most the same number of generators in $L^2(G)$. Also, we yield an extension of the given Bessel sequence to a pair of dual frame sequences.

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1. Introduction

A shift-invariant space (SIS) is a closed subspace V of $L^2(\mathbb{R})$ that is invariant under translations by integers i.e. if $f \in V$, then $f(\cdot+k) \in V$ for all $k \in \mathbb{Z}$. Shift-invariant spaces have many applications in frame theory, numerical analysis, multiresolution analysis(MRA) and wavelet theory [2, 3, 10, 11, 18] The theory of shift-invariant spaces in $L^2(\mathbb{R})$ was presented by Helson [14] and characterized in $L^2(\mathbb{R}^n)$ by Bownik in [4]. In [16, 17] principal shift-invariant spaces, that is, shift-invariant spaces generated by one single function, was studied in the context of locally compact abelian groups. They studied LCA groups with a uniform lattice (i.e. a discrete and co-compact subgroup). The authors in [1]

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presented and investigated shift-invariant spaces for local fields. They studied shift-invariant spaces of $L^2(G)$, where G is a locally compact abelian group, or in general a local field, with a compact open subgroup.

In this paper, we assume G is a locally compact abelian group with uniform lattice H . Then G/H is compact, $H^\perp \approx \widehat{G/H}$ and $\widehat{G}/H^\perp \approx \widehat{H}$, where $H^\perp = \{\xi \in \widehat{G} : \langle \xi, h \rangle = 1, \forall h \in H\}$ is called the annihilator of H and \widehat{G} is dual group of G . In the same manner, all the result of this paper satisfy for local fields.

On the other hand, frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [13] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies et al. [12] and popularized from then on. The theory of frames plays an important role in signal processing, sampling theory and neural networks [6, 7, 8]. For basic results on frames, see [9].

Let H be a Hilbert space and I a set which is finite or countable. A system $\{f_i\}_{i \in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

for all $f \in H$. The system $\{f_i\}_{i \in I} \subseteq H$ is a Bessel sequence if at least the right inequality is satisfied. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame if $A = B$ it is called a Parseval frame. A dual frame for the frame $\{f_i\}_{i \in I}$ is a Bessel sequence $\{g_i\}_{i \in I} \subseteq H$ such that

$$f = \sum_{i \in I} \langle f, g_i \rangle f_i$$

for all $f \in H$.

In [5] was shown that any shift-invariant Bessel sequence in $L^2(\mathbb{R})$ with an at most countable number of generators can be extended to a tight frame by adding another shift-invariant system with at most the same number of generators. Here, we generalize this approach to locally compact abelian groups with a uniform lattice. Similarly, we can do it for local fields.

2. Main Results

First we need the following lemma.

Lemma 2.1. [16] *Let $\{g_n\}_{n \in I}$ be an at most countable family of functions in $L^2(G)$ and $V = \overline{\text{span}}\{T_k g_n\}_{k \in H, n \in I}$. Then there exists a collection of functions $\{\varphi_n\}_{n \in I}$ in $L^2(G)$ such that $\{T_k \varphi_n\}_{k \in H, n \in I}$ is a Parseval frame for V .*

Proposition 2.2. *Let $\{g_n\}_{n \in I}$ be an at most countable family of functions in $L^2(G)$ such that $\{T_k g_n\}_{k \in H, n \in I}$ is a Bessel sequence with bound $B > 0$. Then there exists a collection of functions $\{h_n\}_{n \in I}$ in V such that*

$$\{T_k g_n\}_{k \in H, n \in I} \cup \{T_k h_n\}_{k \in H, n \in I}$$

is a tight frame for V with bound B .

Proof. Consider the linear operator

$$S_1 : L^2(G) \rightarrow L^2(G), S_1 f := \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n.$$

Then S_1 is bounded and the operator

$$S_2 : V \rightarrow V, S_2 := BI - S_1$$

is positive. Thus, S_2 has a unique positive square root, to be denoted by

$S_2^{1/2}$. We can write any $f \in V$ as

$$f = \frac{1}{B} [S_1 f + (BI - S_1) f] = \frac{1}{B} [S_1 f + S_2^{1/2} S_2^{1/2} f].$$

Using the frame decomposition associated with the frame sequence $\{T_k \varphi_n\}_{k \in H, n \in I}$ from Lemma 1.1 (on the element $S_2^{1/2} f$) we arrive at

$$\begin{aligned} f &= \frac{1}{B} \left[S_1 f + S_2^{1/2} \sum_{k \in H, n \in I} \langle S_2^{1/2} f, T_k \varphi_n \rangle T_k \varphi_n \right] \\ &= \frac{1}{B} \left[\sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, S_2^{1/2} T_k \varphi_n \rangle S_2^{1/2} T_k \varphi_n \right]. \end{aligned}$$

It is a standard result that the operator S_2 commutes with the operators T_k . Since $S_2^{1/2}$ is a limit of polynomials in S_2 in the strong operator topology, it follows that also $S_2^{1/2}$ commutes with T_k . Thus, for any $f \in V$ we arrive at

$$f = \frac{1}{B} \left[\sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, T_k S_2^{1/2} \varphi_n \rangle T_k S_2^{1/2} \varphi_n \right]$$

Letting $h_n := S_2^{1/2} \varphi_n \in V$, it follows that

$$\|f\|^2 = \langle f, f \rangle = \frac{1}{B} \left[\sum_{k \in H, n \in I} |\langle f, T_k g_n \rangle|^2 + \sum_{k \in H, n \in I} |\langle f, T_k h_n \rangle|^2 \right].$$

This completes the proof. \square

Example 2.3. Let

$$G = \{(x_j)_{j \in \mathbb{Z}} : x_j \in \{0, 1\} \text{ for all } j \text{ and } x_j = 0 \text{ for all } j > n, \text{ for some } n \in \mathbb{Z}\}.$$

If addition on G is the coordinate-wise addition modulo 2 i.e.

$$(z_j) = (x_j) \oplus (y_j) \Leftrightarrow z_j = x_j + y_j \pmod{2},$$

for all $(x_j), (y_j) \in G$, then G is a locally compact abelian group under Cartesian product topology. The locally compact abelian group G is called Cantor dyadic group (see [15]).

The group G is identified with the half real line $[0, \infty)$ under the mapping $x \rightarrow |x| = \sum_{j \in \mathbb{Z}} x_j 2^j$.

Let

$$H = \{x \in G : x_j = 0, \text{ for } j < 0\} \text{ and } D = \{x \in G : x_j = 0, \text{ for } j \geq 0\}.$$

Then H is a uniform lattice, D is compact and $G/H = D$. Under the map $x \rightarrow |x|$, H is identified with the nonnegative integers and D is identified with the interval $[0, 1]$.

Now, let the system $\{T_k \chi_D : k \in H\}$. Then

1. The system $\{T_k \chi_D + T_{k \oplus 1} \chi_D : k \in H\}$ is a Bessel sequence in $L^2(G)$ with bound $B = 4$;
2. $\{T_k \chi_D + T_{k \oplus 1} \chi_D : k \in H\}$ is not a frame sequence;
3. $V := \overline{\text{span}}\{T_k \chi_D + T_{k \oplus 1} \chi_D : k \in H\} = \overline{\text{span}}\{T_k \chi_D : k \in H\}$

In the same manner of Example 2.3 in [7], if $h = \chi_D - T_1 \chi_D$, then

$$\{T_k \chi_D + T_{k \oplus 1} \chi_D : k \in H\} \cup \{T_k h : k \in H\}$$

is a tight frame for V with bound 4.

Note: Similar to Example 2.5 in [7] we can show the generators $\{h_n\}_{n \in \mathbb{Z}}$ in proposition 2.1 is optimal.

Corollary 2.4. Let $\{g_n\}_{n \in I}$ be an at most countable family of functions in $L^2(G)$ such that $\{T_k g_n\}_{k \in H, n \in I}$ is a Bessel sequence with bound $B > 0$. Then there exists functions $\{\varphi_n\}_{n \in I}$ and $\{\psi_n\}_{n \in I}$ in V such that

$$\{T_k g_n\}_{k \in H, n \in I} \cup \{T_k \varphi_n\}_{k \in H, n \in I}, \{T_k g_n\}_{k \in H, n \in I} \cup \{T_k \psi_n\}_{k \in H, n \in I}$$

form a pair of dual frames for V .

Proof. Let

$$S_1 : L^2(G) \rightarrow L^2(G), S_1 f := \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n.$$

For any $f \in V$, applying the frame decomposition associated with the frame sequence $\{T_k \varphi_n\}_{k \in H, n \in I}$ from Lemma 1.1, we can write f as follows:

$$f = S_1 f + (I - S_1) f \tag{1}$$

$$= \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle (I - S_1) f, T_k \varphi_n \rangle T_k \varphi_n \tag{2}$$

$$= \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, (I - S_1)^* T_k \varphi_n \rangle T_k \varphi_n. \tag{3}$$

It's clear that $(I - S_1)^* = (I - S_1)$. Now, for any $h \in V$, from Lemma 7.21 in [9] we have

$$(I - S_1) T_k h = T_k h - S_1 T_k h = T_k h - T_k S_1 h = T_k (I - S_1) h.$$

Hence

$$(I - S_1) T_k = T_k (I - S_1).$$

Put $\psi_n = (I - S_1) \varphi_n$, Then (1.1) becomes

$$\begin{aligned} f &= \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, (I - S_1) T_k \varphi_n \rangle T_k \varphi_n \\ &= \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, T_k (I - S_1) \varphi_n \rangle T_k \varphi_n \\ &= \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n + \sum_{k \in H, n \in I} \langle f, T_k \psi_n \rangle T_k \varphi_n. \end{aligned}$$

This proves that

$$\{T_k g_n\}_{k \in H, n \in I} \cup \{T_k \varphi_n\}_{k \in H, n \in I} \text{ and } \{T_k g_n\}_{k \in H, n \in I} \cup \{T_k \psi_n\}_{k \in H, n \in I}$$

form a pair of dual frames for V . \square

Theorem 2.5. Suppose that $\{T_k g_n\}_{k \in H, n \in I}$ is a Bessel sequence in $L^2(G)$ with the upper bound B . For any $\lambda \geq B$, there is a sequence $\{h_m\}_{m \in \mathbb{Z}}$ such that $\{T_k g_n\}_{k \in H, n \in I} \cup \{T_k h_m\}_{k \in H, m \in \mathbb{Z}}$ is a tight frame for $L^2(G)$ with the bound λ .

Proof. Let

$$S_1 f = \sum_{k \in H, n \in I} \langle f, T_k g_n \rangle T_k g_n, \quad \forall f \in L^2(G).$$

Since $\{T_k g_n\}_{k \in H, n \in I}$ is a Bessel sequence in $L^2(G)$ with the upper bound B , S_1 is self-adjoint, positive and satisfies

$$\langle Sf, f \rangle \leq B \|f\|^2, \quad \forall f \in L^2(G).$$

For any $\lambda \geq B$, define $S_2 : L^2(G) \rightarrow L^2(G)$ by

$$S_2 = \lambda I - S_1.$$

Then S_2 is positive and commutes with T_n . From Lemma 2.4.4 in [9] we know that $S_2^{1/2}$ exists, is self-adjoint and can be expressed as a limit of a sequence of polynomials in S_2 . Accordingly $S_2^{1/2}$ commutes with T_n .

Take $\{T_k \psi_m\}_{k \in H, m \in \mathbb{Z}}$ to be a Parseval frame for $L^2(G)$. Like in the proof of Proposition 2.2 we see that

$$\begin{aligned} S_2 f &= S_2^{1/2} S_2^{1/2} f = S_2^{1/2} \sum_{k \in H, m \in \mathbb{Z}} \langle S_2^{1/2} f, T_k \psi_m \rangle T_k \psi_m \\ &= \sum_{k \in H, m \in \mathbb{Z}} \langle f, S_2^{1/2} T_k \psi_m \rangle S_2^{1/2} T_k \psi_m \\ &= \sum_{k \in H, m \in \mathbb{Z}} \langle f, T_k S_2^{1/2} \psi_m \rangle T_k S_2^{1/2} \psi_m. \end{aligned}$$

Put $h_m = S_2^{1/2} \psi_m$. Since $S_1 + S_2 = \lambda I$, it follows that

$$\{T_k g_n\}_{k \in H, n \in I} \cup \{T_k h_m\}_{k \in H, m \in \mathbb{Z}}$$

is a tight frame for $L^2(G)$ with frame bound λ . \square

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