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Solving a Nonlinear Multi-Order Fractional Differential Equation Using Haar wavelets

F. Ghomanjani*

Abstract

In this paper, the Haar wavelets is applied to obtain approximate solutions of nonlinear multi-order fractional differential equations (M-FDEs). The fractional derivative is described in the Caputo sense. Numerical example is presented to verify the efficiency and accuracy of the proposed algorithm. The results reveal that the method is accurate and easy to implement.

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1. Introduction

Many phenomena in engineering physics, chemistry, and other sciences can be described very successfully by models that use mathematical tools of fractional calculus, i.e. the theory of derivatives and integrals of non-integer order. For example, they have been successfully used in modeling frequency dependent damping behavior of many viscoelastic materials. There are numerous research which demonstrate the applications of fractional derivatives in the areas of electrochemical processes, dielectric polarization, colored noise, and chaos [1].

The organization of this study is arranged as follows. In Section 2, Basic Preliminaries is presented. Properties of the Haar basis is stated in 3. An numerical example is provided in Section 4. Finally, Section 5 will give a conclusion briefly.

2. Basic Preliminaries

Let $x : [a, b] \to \mathcal{R}$ be a function, $\alpha > 0$ a real number, and $n = \alpha$, where α denotes the smallest integer greater than or equal to α (see [2]). The left (left RLFI) and right (right RLFI) Riemann-Liouville fractional integrals are defined by

$$ {}_aI_t^{\alpha}\, x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau, \quad \text{(left RLFI)},$$

$$ {}_tI_b^{\alpha}\, x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau)\, d\tau, \quad \text{(right RLFI)},$$

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$$\begin{aligned} {}_aD_t^{\alpha}\, x(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)\, d\tau, \quad \text{(left RLFD)},\\ {}_tD_b^{\alpha}\, x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1}x(\tau)\, d\tau, \quad \text{(right RLFD)}, \end{aligned} \tag{1}$$

Moreover, the left (left CFD) and right (right CFD) Caputo fractional derivatives are defined by means of

$$\begin{aligned} {}_a^CD_t^{\alpha}\, x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)\, d\tau, \quad \text{(left CFD)},\\ {}_t^CD_b^{\alpha}\, x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}x^{(n)}(\tau)\, d\tau, \quad \text{(right CFD)}, \end{aligned} \tag{2}$$

The relation between the right RLFD and the right CFD is as follows:

$$ {}_t^CD_b^{\alpha}\, x(t) = {}_tD_b^{\alpha}\, x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, \tag{3}$$

Further, it holds

$$ {}_0^CD_t^{\alpha}\, c = 0, \tag{4}$$

where c is a constant, and

$$ {}_0^CD_t^{\alpha}\, t^n = \begin{cases} 0, \text{for } n \in \mathbb{N}_0, \text{ and } n < \lceil \alpha \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil \end{cases} \tag{5}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We recall that for $\alpha \in N$ the Caputo differential operator coincides with the usual differential operator of integer order.
In this paper, fractional differential equation was considered

$$\begin{aligned} &D^{\alpha}x(t) = a(t)x(t) + \sum_{r=1}^{l} b_r(t)D^{\alpha_r}x(q_r t), \; m-1 < \alpha \leq m, \; t \in [0, b],\\ &x^{(i)}(t) = \mu_i, i = 0, 1, \ldots, m-1. \end{aligned}$$

Here, $0 < q_r < 1$, $0 \leq \alpha_r < \alpha \leq m$, $r = 1, 2, \ldots, l$; x is an unknown function; a and b_r, $r = 1, 2, \ldots, l$, are the known functions defined in $[0, b]$.

3. Properties of the Haar basis

The RH functions $RH(r, t)$, $r = 1, 2, \ldots$, are composed of three values $+1$, -1, 0 and can be defined on the interval $[0, 1)$ by [2] as

$$RH(r,t) = \begin{bmatrix} 1, & J_1 \leq t \leq J_{\frac{1}{2}} \\ -1, & J_{\frac{1}{2}} \leq t \leq J_0 \\ 0, & otherwise \end{bmatrix}$$

where

$$J_u = \frac{j-u}{2^i}, \ u = 0, \frac{1}{2}, 1. \tag{6}$$

The value of r defines two parameters i and j via

$$r = 2^i + j - 1, \ i = 0, 1, 2, 3, \ldots, \ j = 1, 2, 3, \ldots, 2^i.$$

$RH(0, t)$ is defined for $i = j = 0$ and is given by

$$RH(0, t) = 1, \ 0 \le t \le 1. \tag{7}$$

A set of the there RH functions is shown in Figs. 1,2,3, where, $r = 3, 4, 5$. A set of there RH functions is shown in Figs. 1,2,3, where, $r = 3, 4, 5$. The orthogonality property is given by

$$\int_0^1 RH(r,t)RH(v,t)\,dt = \left[\begin{array}{ll} 2^{-i}, & r = v \\ 0, & r \neq v \end{array} \right.$$

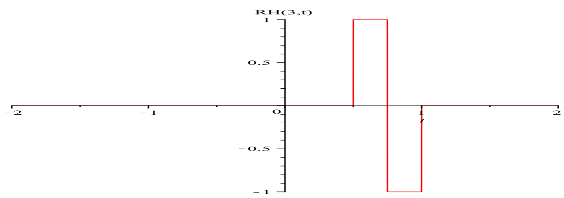

Figure 1. The graph of RH function

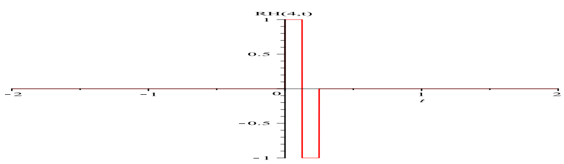

Figure 2. The graph of RH function

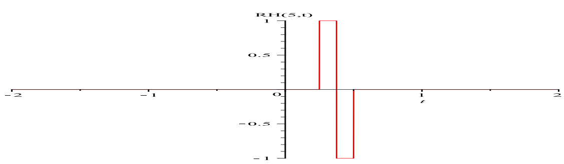

Figure 3. The graph of RH function

4. Numerical example

In this section, a numerical example is presented to illustrate the proposed method.

Example 4.1. *Consider the fractional neutral pantograph differential equation (see [2])*

$$D^{\gamma}x(t) = \frac{3}{4}x(t) + x(\frac{1}{2}t) + D^{\gamma_1}x(\frac{1}{2}t) + \frac{1}{2}D^{\gamma}x(\frac{1}{2}t) - t^2 - t + 1,\ 0 < \gamma_1 < \gamma \le 2,$$
$$x(0) = x'(0) = 0$$

TABLE 1. The comparision of the absolute errors on interval [0, 1] for Example 4.1

t	Runge-Kutta method	Present method
0.1	1×1^{-3}	3.4×10^{-6}
0.2	2.02×10^{-3}	5.1×10^{-5}
0.3	3.07×10^{-3}	3.2×10^{-5}
0.4	3.14×10^{-3}	5.8×10^{-4}
0.5	5.34×10^{-3}	4.1×10^{-4}

5. Conclusions

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason one may need a reliable and efficient technique for the solution of fractional differential equations. This paper deals with the approximate solution of a class of multi-order fractional differential equations. The fractional derivatives are described in the Caputo sense. Our main aim is to evaluation of fractional derivative using Haar wavelet collocation method and implementing it to solve the nonlinear multi-order fractional differential equations. Illustrative example is included to demonstrate the validity and applicability of the technique.

References

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F. GHOMANJANI,
Kashmar Higher Education Institute, Kashmar, Iran.
e-mail: fatemeghomanjani@gmail.com