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Word
On harmonic vector fields on Finsler manifolds

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Abstract

In the present work, we try to extend the definition of harmonic vector fields on Finsler manifolds. For aiming this purpose, we define suitable Dirichlet energy and introduce harmonic vector fields as the critical points of defining action. Then we extend the Hodge de Rahm harmonic vector fields on m- Finsler manifolds and Finally we compare these two kinds of definitions of harmonic vector fields with each other.

Keywords: Harmonic vector field, Dirichlet energy, Hodge-de Rahm harmonic.

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1 Introduction

There are two different approaches to definitions of harmonic vector fields in Riemannian geometry and each of which has its own application. The Hodge-de Rham operator, $\Delta_H = d\delta + \delta d$, works in the space of forms but it is possible to consider a vector field as a form of using the musical morphism. Through this way, Bochner [2] uses the Hodge-de Rham operator to prove some relations between harmonic vector fields and the curvature of Riemannian manifolds. In [6], there is a complete monograph on harmonic vector fields that arises as the critical points of the Dirichlet energy. When the domain of the Dirichlet action is restricted to the space of unit tangent vector fields, the harmonic vector field appears as an eigenvector of a suitable PDE system. This method is useful for the study of contact geometry and nonlinear sub-elliptic systems. Clearly, the Euler-Lagrange equation of harmonic vector fields is not similar to the Euler-Lagrange equation of harmonic maps in the variational approach unless the additional curvature condition is satisfied. It is well known that the definition of the harmonic vector fields based on the Hodge-de Rham operator is equivalent to the definition of unit harmonic vector fields if and only if $(M, g)$ is an Einstein manifold, see [6].

The concept of a harmonic map on Finsler manifolds is studied in different papers. They use a variational approach to defining a harmonic map. Unfortunately, this method can not be extended to define a harmonic vector field on Finsler manifolds because of needing to lift vector fields on $TM$. In [3, 4], the authors used the Bochner technique by lifting vector fields to define harmonic vector fields on Finsler manifolds. They obtained a necessary and sufficient condition for a lifted vector field to be harmonic on a Finsler manifold and

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particularly on Landsberg manifolds. Since there are various definitions of volume forms in Finsler geometry, there is not a unique definition of the Finsler Laplacian. Moreover, in this kind of definitions, all Laplacians are linear elliptic operators. In [7], Ohta and Strum introduced a new action to define the harmonic maps through a variational approach on Finsler measure spaces. By candicating Finsler m spaces, the Laplacian of functions on M is a non-linear elliptic operator on an underlying manifold with respect to any measure.

Our main purpose of this work is to define harmonic vector fields through Bochner technique and variational approach on Finsler m spaces and then compare these two kinds of definitions. Hence, we use the Bochner technique on Finsler m spaces and derive a PDE system to define σ-harmonic vector fields on Finsler manifolds.

2 Main results

A Finsler structure $F$ on a manifold $M$ is a Minkowski norm, that is, it is a smooth function on the tangent bundle $TM$ satisfying the following conditions:

- $F$ is a smooth function on the entire slit tangent bundle $TM_0 := TM - \{0\}$.
- $F$ is a positive homogeneous function on the second variable, $y$.
- The matrix $(g_{ij})_{1 \leq i,j \leq n}$, $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y_i \partial y_j} F^2(x, y)$ is positive definite.

One can consider a collection of weighted Riemannian manifolds at every point of a Finsler manifold. We know that a weighted Riemannian manifold is a triple $(M, g, f)$ with $(M, g)$ a Riemannian manifold and $f$ a smooth and strictly positive function on $M$. Let us denote the usual volume form associated to the Riemannian metric $g$ by $dv_g$. We consider the new measure on $(M, g)$ by $dm = f dv_g$. On any weighted Riemannian manifold, there exists an induced divergence of vector fields $X \in \Gamma(TM)$,

$$\text{div}_m X = \frac{1}{f} \text{div}(f X),$$

and the corresponding weighted Laplacian

$$\Delta_m = \text{div}_m o \nabla,$$

where $\nabla$ is the gradient operator on smooth function space $C^\infty(M)$. To see such a kind of geometric structure on Finsler manifolds, it is defined $\hat{g}(x) := g_Y(x) = g(x, Y(x))$ a Riemannian metric structure on $M$, for all non-vanishing vector field $Y$ on $M$, [5]. There are also two different volume forms on Finsler manifolds. Akbar-Zadeh in [1] used the Hilbert form, $\omega(x, y) := \frac{\partial}{\partial y} F(x, y) dx^i$ to define a volume form on the indicatrix bundle $SM := \{(x, y) \in TM_0 | F(x, y) = 1\}$ by

$$\eta = \frac{(-1)^{n(n-1)} x}{(n-1)!} \omega \wedge (d\omega)^{n-1}.$$
Let \((\omega_i)_{1 \leq i \leq n}\) be the adapted frame fields on the dual Finsler bundle and \(\omega_{ij}\) the Chern connection form. Hence \(\eta\) is written in the form \(\eta := \omega_1 \wedge \cdots \wedge \omega_n \wedge \omega_{n,1} \wedge \cdots \wedge \omega_{n,n-1}\).

On the other hand, for any real valued function \(f\) on the indicatrix bundle \(SM\), we have
\[
\int_{SM} f\eta = \int_M dx \int_{S_r M} f \sqrt{\det g_{ij} \omega_{n,1} \wedge \cdots \wedge \omega_{n,n-1}}.
\]
where \(dx = dx^1 \wedge \cdots \wedge dx^n\). Set \(\alpha(x) = \int_{S_r M} \sqrt{\det g_{ij} \omega_{n,1} \wedge \cdots \wedge \omega_{n,n-1}}\) which is a positive function on \(M\). Hence it is deduced that \(dm := \alpha(x) dx\) is a Borel measure on a compact Finsler manifold \(M\). Therefore, we can consider a collection of weighted Riemannian manifolds \((M, \hat{g}, m)\) at each point of the Finsler manifold \((M, F)\). The weighted gradient vector field on a weighted Riemannian manifold \((M, \hat{g}, m)\) is defined by
\[
\nabla^Y u(x) = g^{ij}(x, Y) \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j},
\]
in local coordinate system \((x^i)\) then the weighted Laplacian on \((M, \hat{g}, m)\) is
\[
\Delta^Y_m u := \text{div}_m(\nabla^Y u),
\]
where \(m\) is a Borel measure. It is easy to see that a compact Finsler space with a volume form on \(SM\) admits a regular metric measure space structure denoted here by \((M, F, m)\) and called Finsler m space, c.f., [5]. It is sufficient to consider the metric structure by
\[
d(x_i, x_j) := \inf_{\gamma} \left( \int_0^1 (F^2(\gamma(t), \dot{\gamma}(t)) \right)^{\frac{1}{2}} dt, \quad \forall x_i, x_j \in M,
\]
where \(\gamma\) is an arbitrary smooth curve on a manifold \(M\) such that \(\gamma(0) = x_i\) and \(\gamma(1) = x_j\). The Legendre transform \(l : TM \to T^*M\) is a smooth map defined by \(l(y) := g_y(y, .)\) for all \(y \in T_x M\) on Finsler space \((M, F)\). For a differentiable function \(u : M \to \mathbb{R}\), the gradient vector field is defined by
\[
\nabla u := l^{-1}(x, Du),
\]
where \(Du(x) = \frac{\partial u}{\partial x^i}(x) dx^i\). Hence in local coordinate system, we have
\[
\nabla u = g^{ij}(x, Du(x)) \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}.
\]
The Laplacian \(\Delta u\) on the space of compact support smooth functions is defined through the identity
\[
\int_\Omega v \Delta u dm = \int_\Omega (\text{div} u)v dm = - \int_\Omega D u(\nabla u) dm,
\]
where \(\Omega\) is a domain in \(M\) and \(v \in C_c^\infty(M)\). Ohta and Strum in [7] defined the Finsler Laplacian on the space of smooth functions by
\[
\Delta u := \text{div} (\nabla u) = \text{div}(l^{-1}(Du)).
\]
It is proved that the Finsler Laplacian of smooth functions coincides with the weighted Laplacian in the direction $\nabla u$ in the sense of distributions on $\Omega$, that is

$$\Delta u = \Delta_m^\nabla u,$$

see Lemma 2.4 in [7].

The Riemann curvature of a Finsler space is a family of linear transformations on tangent spaces, given by

$$R = \{ R_y : T_x M \to T_x M | y \in T_x M_0, x \in M \},$$

with components

$$R^i_{jk}(x, y) = 2 \frac{\partial G^i}{\partial x^j}(x, y) y^j - \frac{\partial G^i}{\partial y^j}(x, y) \frac{\partial G^j}{\partial y^k}(x, y) + \frac{\partial G^i}{\partial y^j}(x, y) \frac{\partial G^j}{\partial x^k}(x, y),$$

derived from the variations of geodesics. The geodesic spray of the Finsler structure $F$ is defined by

$$G(x, y) = y^i(x) \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i(x, y) = \gamma^i_{jk}(x, y)y^j y^k$, $\gamma^i_{jk}$ are the Christoffel symbols of the fundamental tensor and $y \in T_x M$. Constant speed geodesics are curves on $M$ that satisfy the second order system of ODEs

$$\frac{d^2}{dt^2} \gamma^i + G^i(\gamma, \dot{\gamma}) = 0.$$ 

Let $\gamma(t)$ be a smooth curve in $M$, with velocity $\dot{\gamma}(t) := \frac{d}{dt}\gamma(t)$. Its canonical lift into the slit tangent bundle $TM_0$ is defined by $\hat{\gamma}(t) := (\gamma(t), \dot{\gamma}(t))$. If $\gamma(t)$ is a constant speed geodesic then its canonical lift $\hat{\gamma}(t)$ is an integral curve of $G$. Let $Y$ be a geodesic vector field on an open subset $U$ in a Finsler space $(M, F)$. It is proved in [5], if $R_y$ is the Riemann curvature of $F$ and $\hat{R}_y$ is the Riemann curvature of $\hat{F} := \sqrt{y}$ then $R_y = \hat{R}_y$, where $y = Y_x \in T_x M$. Define the trace of Riemann curvature by $\text{Ric}(x, y) = R^i_i(x, y)$, which is called the Ricci curvature. The Ricci scalar is $R(x, y) := \frac{1}{n-1} \text{Ric}(x, y)$. A Finsler manifold is called a generalized Einstein manifold if the Ricci curvature is independent of the direction, see [1].

**Theorem 2.1.** Let $M$ be a closed Finsler m space and $X$ be a section of $M$. If

$$\begin{cases}
J := (\Delta_m^X X_i - X^k R_{ki}) X^i \geq 0, \\
I := (2X^i \nabla_i X_k - \delta(t(X)) X_k - X^i \nabla_k X_i) \nabla^k \sigma \geq 0,
\end{cases}$$

then $X$ is a Hodge-de Rahm harmonic vector field.

We prove that the only non-trivial solution of this sub-system occurs in the equality case. Then we define the Dirichlet action of vector fields on associated weighted Riemannian manifolds of a Finsler m space. It is used to define the Finsler Laplaceian of vector fields on a closed Finsler m space. Let $X : M \to TM$ be a
smooth section of closed Finsler m space $M$. For a given $x \in M$ and $\xi_x \in T_xM$, let $Z$ be a geodesic smooth vector field on $U \subset M$ with $Z(x) = \xi_x$. Since $g_Z(x) := g(x, \xi_x)$ is a Riemannian structure on $U$, we consider its associated Riemann-Sasaki metric as follows

$$G = (g_Z)_{ij} dx^i \otimes dx^j + (g_Z)_{ij} dy^i \otimes dy^j.$$ 

Define the energy density of map $X$ by

$$e(X) : M \to \mathbb{R}^+ \cup \{0\},$$

$$e(X)(x) = \frac{1}{2} ||dX||^2,$$

$$= \frac{1}{2} G (dX(\frac{\partial}{\partial x^i}), dX(\frac{\partial}{\partial x^j})), $$

hence, the energy of map $X$ is

$$E(X) := \int_M e(X) dm.$$  \hspace{1cm} (2.2)

**Proposition 2.2.** The energy of the map $X$ can be splitted into horizontal and vertical components

$$E(X) = E^h(X) + E^v(X),$$

where $E^h(X) = n/2$ and $E^v(X) = \frac{1}{2} \int_M || \nabla X ||^2 dm.$

**Definition 2.3.** A vector field $X$ is a harmonic vector field on a closed Finsler m space if it is the stationary point of the action $E$, i.e.,

$$\frac{d}{dt} \bigg|_{t=0} E(X_t) = 0,$$

or equivalently

$$\frac{d}{dt} \bigg|_{t=0} E^v(X_t) = 0.$$ 

**Theorem 2.4.** A vector field $X$ is a $\sigma$-harmonic vector field if and only if

$$\Delta X = \lambda X,$$  \hspace{1cm} (2.3)

where $\lambda$ is constant.

In general, a Hodge-de Rahm harmonic vector field is not necessarily a $\sigma$-harmonic vector field. The next theorem shows when it has happened.

**Theorem 2.5.** Let $M$ be a closed Finsler m space. Suppose that $X$ is a unit Hodge-de Rahm harmonic vector field. If $(M, F)$ is a generalized Einstein manifold then $X$ is a $\sigma$-harmonic vector field, as well.
References


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