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The restrained \(k\)-rainbow reinforcement numbers in graphs

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Abstract

For a positive integer \(k\), a restrained \(k\)-rainbow dominating function (R\(k\)RDF) of a graph \(G\) is a function \(f\) from the vertex set \(V(G)\) to the set of all subsets of the set \(\{1,2,\ldots,k\}\) such that for any vertex \(v \in V(G)\) with \(f(v) = \emptyset\) the conditions \(\bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}\) and \(|N(v) \cap \{u \in V(G) \mid f(u) = \emptyset\}| \geq 1\) are fulfilled, where \(N(v)\) is the open neighborhood of \(v\). The weight of an R\(k\)RDF \(f\) is the value \(\omega(f) = \sum_{v \in V(G)} |f(v)|\). The restrained \(k\)-rainbow domination number of a graph \(G\), denoted by \(\gamma_{rrk}(G)\), is the minimum weight of an R\(k\)RDF of \(G\). The restrained \(k\)-rainbow reinforcement number \(r_{rrk}(G)\) of a graph \(G\) is the minimum number of edges that must be added to \(G\) in order to decrease the restrained \(k\)-rainbow domination number. In this paper, we initiate the study of restrained \(k\)-rainbow reinforcement number in graphs and we present some sharp bounds on \(r_{rrk}(G)\). In particular, we determine the restrained \(2\)-rainbow reinforcement number of some classes of graphs.

Keywords: Retrained \(k\)-rainbow domination number, restrained \(k\)-rainbow reinforcement number.

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1 Introduction

In this paper, \(G\) is a simple graph with vertex set \(V = V(G)\) and edge set \(E = E(G)\). The order \(|V|\) of \(G\) is denoted by \(n = n(G)\). For every vertex \(v \in V(G)\), the open neighborhood \(N_G(v) = N(v)\) is the set \(\{u \in V(G) \mid uv \in E(G)\}\) and the closed neighborhood of \(v\) is the set \(N_G[v] = N[v] = N(v) \cup \{v\}\). The degree of a vertex \(v \in V\) is \(\deg_G(v) = \deg(v) = |N(v)|\).

The minimum and maximum degree of a graph \(G\) are denoted by \(\delta = \delta(G)\) and \(\Delta = \Delta(G)\), respectively. We write \(K_{n,m}\) for the complete bipartite graph of order \(n+m\), \(C_n\) for a cycle of length \(n\) and \(P_n\) for a path of order \(n\).

A subset \(S\) of vertices of \(G\) is a dominating set if \(N[S] = V\). The domination number \(\gamma(G)\) is the minimum cardinality of a dominating set of \(G\). A dominating set of minimum cardinality of \(G\) is called a \(\gamma(G)\)-set. The reinforcement number \(r(G)\) of a graph \(G\) is the minimum number of edges that must be added to \(G\) in order to decrease the domination number [13]. The reinforcement number is defined to be 0 when \(\gamma(G) = 1\).

For a positive integer \(k\), a restrained \(k\)-rainbow dominating function (R\(k\)RDF) of a graph \(G\) is a function \(f\) from the vertex set \(V(G)\) to the set of all subsets of the set \(\{1,2,\ldots,k\}\).
such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ and $|N(v) \cap \{u \in V(G) \mid f(u) = \emptyset\}| \geq 1$ are fulfilled. The weight of a R$k$ RDF $f$ is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The restrained $k$-rainbow domination number of a graph $G$, denoted by $\gamma_{rrk}(G)$, is the minimum weight of a R$k$ RDF of $G$. A $\gamma_{rrk}(G)$-function is a restrained $k$-rainbow dominating function of $G$ with weight $\gamma_{rrk}(G)$. Note that $\gamma_{rr1}(G)$ is the classical restrained domination number $\gamma_r(G)$. The $k$-rainbow domination number was introduced by Brešar, Henning, and Rall [4] and has been studied by several authors (see for example [5, 6, 11]).

Our purpose in this paper is to initiate the study of restrained $k$-rainbow reinforcement number in graphs. We determine exact values of restrained 2-rainbow reinforcement number of some classes of graphs.

2 Main result

We will use the following results.

**Proposition 2.1.** If $E$ is a $r_{rrk}(G)$-set, then
$$\gamma_{rrk}(G) - 2 \leq \gamma_{rrk}(G + E) \leq \gamma_{rrk}(G) - 1.$$  

**Proposition 2.2.** $\max\{\gamma_r(G), \gamma_r(G)\} \leq \gamma_{rrk}(G) \leq k \gamma_r(G)$.

**Proposition 2.3.** [1] For $n \geq 4$, $\gamma_{rr2}(P_n) = \lceil \frac{2n+1}{3} \rceil + 1$ and $\gamma_{rr2}(P_n) = n$, otherwise.

**Proposition 2.4.** [1] For $n \geq 6$,
$$\gamma_{rr2}(C_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1 & n \equiv 2 \pmod{3} \\ 2\lceil \frac{n}{3} \rceil & \text{otherwise}. \end{cases}$$

**Proposition 2.5.** [2] For $1 \leq n \leq m$,
$$\gamma_{rr2}(K_{n,m}) = \begin{cases} m + 1 & n = 1 \\ 4 & n \geq 2. \end{cases}$$

**Theorem 2.6.** For $n \geq 3$, $r_{rr2}(P_n) = 1$

*Proof.* Let $P_n := v_1v_2 \ldots v_n$. If $3 \leq n \leq 6$, then it is not hard to see that $r_{rr2}(P_n) = 1$. So suppose that $n \geq 7$. We consider three cases.

**Case 1.** $n \equiv 1 \pmod{3}$.

Define $f : V(P_n) \to P(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f \cup v_n, \emptyset, V_{1,2}^f)$ is an R2 RDF on $P_n + v_1v_{n-1}$ that implies $r_{rr2}(P_n) = 1$.

**Case 2.** $n \equiv 2 \pmod{3}$.

Define $f : V(P_n) \to P(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$, $f(v_n) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f \cup v_n, \emptyset, V_{1,2}^f)$ is an R2 RDF on $P_n + v_2v_n$ that implies $r_{rr2}(P_n) = 1$.

**Case 3.** $n \equiv 0 \pmod{3}$.

Define $f : V(P_n) \to P(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$, $f(v_n) = f(v_{n-1}) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f \cup \{v_{n-1}, v_n\}, V_{1,2}^f - \{v_{n-1}, v_n\}, \emptyset, V_{1,2}^f)$ is an R2 RDF on $P_n + v_1v_n$ that implies $r_{rr2}(P_n) = 1$. \qed
Theorem 2.7. For $n \geq 6$,
\[ r_{rr2}(C_n) = \begin{cases} 
2 & n \equiv 0 \pmod{3} \\
1 & \text{otherwise}.
\end{cases} \]

Proof. Let $C_n := (v_1, v_2, \ldots, v_n)$. We consider three cases.

Case 1. $n \equiv 0 \pmod{3}$.

Define $f : V(C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(C_n)$-function of weight $\frac{2n}{3}$. Then the function $g = (V'_0, v_{n-2}, V_{1,2} - v_{n-2})$ is an R2RDF on $C_n + \{v_{n-1}, v_{n-3}\}$, that implies $r_{rr2}(C_n) \leq 2$.

It is not hard to see that for any $\gamma_{rr2}(C_n)$-function $g$, we have $|V'_0| = \frac{n}{3}$ and $|V'_0 \cup V^2_{1,2}| = 0$.

Thus each vertex of $V^{2}_{1,2}$ has exactly two private neighbors. This implies that $r_{rr2}(C_n) \geq 2$.

Therefore $r_{rr2}(C_n) = 2$ in this case.

Case 2. $n \equiv 1 \pmod{3}$.

Define $f : V(C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(C_n)$-function of weight $2\frac{2n}{3}$. Then the function $g = (V'_0, \emptyset, \emptyset, V_{1,2} - v_n)$ is an R2RDF on $C_n + v_n$, that implies $r_{rr2}(C_n) = 1$ in this case.

Case 3. $n \equiv 2 \pmod{3}$.

Define $f : V(C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod{3}$, $f(v_n) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly $f$ is a $\gamma_{rr2}(C_n)$-function of weight $2\frac{2n}{3} + 1$. Then the function $g = (V'_0 \cup \{v_n\}, \emptyset, \emptyset, V_{1,2}^f)$ is an R2RDF on $C_n + v_n$, that implies $r_{rr2}(C_n) = 1$.

\[ \square \]

Theorem 2.8. For $1 \leq n \leq m$,
\[ r_{rr2}(K_{n,m}) = \begin{cases} 
1 & n = 1, 2, m \geq 2 \\
2 & n \geq 3.
\end{cases} \]

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be the partite sets of $K_{n,m}$. We consider three cases.

Case 1. $n = 1, m \geq 2$.

Define function $f : V(K_{n,m}) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(y_i) = \{1\}$ for $1 \leq i \leq m$ and $f(x_1) = \{2\}$. Clearly $f$ is a $\gamma_{rr2}(K_{n,m})$-function of weight $m + 1$. Let $G = K_{n,m} + \{y_1y_2\}$ and define $g : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $g(x_1) = \{1, 2\}$, $g(y_1) = g(y_2) = \emptyset$ and $g(x) = \{1\}$ otherwise. Obviously $g$ is an R2RDF on $G$ of weight $m$. Thus $r_{rr2}(K_{1,m}) = 1$ in this case.

Case 2. $n = 2$.

Define $f : V(K_{n,m}) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(x_1) = f(y_1) = \{1, 2\}$ and $f(x) = \emptyset$ otherwise. Then $f$ is a $\gamma_{rr2}(K_{n,m})$-function of weight $4$. Join $x_1$ to $x_2$ and define $g : V(K_{n,m} + \{x_1x_2\}) \rightarrow \mathcal{P}(\{1, 2\})$ by $g(x_1) = \{1, 2\}$ and $g(x) = \emptyset$ otherwise. Obviously $g$ is an R2RDF on $K_{n,m} + \{x_1x_2\}$ of weight $2$. Therefore $r_{rr2}(K_{2,m}) = 1$.

Case 3. $n \geq 3$. Define $f : V(K_{n,m}) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(x_1) = f(y_1) = \{1, 2\}$ and $f(x) = \emptyset$ otherwise. Let $G = K_{n,m} + \{x_1x_2, x_1x_3, \ldots, x_1x_{n-1}\}$. Define $g : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $g(x_1) = \{1, 2\}$, $g(x_n) = \{1\}$ and $g(x) = \emptyset$ otherwise. Obviously $g$ is an R2RDF on $G$ of weight $3$. So $r_{rr2}(K_{n,m}) \leq n - 2$. Now let $\gamma_{rr2}(K_{n,m} + E) = 3$. Clearly, for an arbitrary graph $G$ of order $p$, $\gamma_{rr2}(G) = 3$ if and only if either $|V_{1,2}^f| = 1$ and $\Delta(G) = p - 2$ or $|V_{1,2}^f| = 0$ and $\Delta(G) \geq p - 3$. Thus either $|E| \geq n - 2$ or $|E| \geq n - 3$. If $|E| \geq n - 2$, we are done. Suppose that $\gamma_{rr2}(K_{n,m} + E) = 3$ and $|E| \geq n - 3$. But in this case, by adding any $n - 3$ edges to $K_{n,m}$ we have $\gamma_{rr2}(K_{n,m} + E) = 4$. So $r_{rr2}(K_{n,m}) = n - 2$. The proof is complete.
References


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