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Some iterative methods for solving an operator equation by using g-frames

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Abstract
This paper proposes some iterative methods for solving an operator equation on a separable Hilbert space $H$ equipped with a g-frame. We design some algorithms based on the Richardson and Chebyshev methods and investigate the convergence and optimality of them.

Keywords: Hilbert space, g-frame, operator equation, iterative method, Chebyshev polynomials.


1 Introduction and preliminaries

G-frames are natural generalization of frames and provide more choices on analyzing functions from frame expansion coefficients. Let $J$ be a countable index set and $\{\Lambda_j\}_{j \in J}$ be a set of operators from a separable Hilbert space $H$ to another separable Hilbert space $V_j$ for $j \in J$. The sequence $\{\Lambda_j\}_{j \in J}$ is called a $g$-frame for $H$ with respect to $\{V_j\}_{j \in J}$ if there are two positive $A$ and $B$ such that

$$A\|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \forall f \in H.$$ 

$A$ and $B$ is called the lower and upper frame bound, respectively. If $A = B$ then $\{\Lambda_j\}_{j \in J}$ is called a tight g-frame. The g-frame operator $S$ for a g-frame $\{\Lambda_j\}_{j \in J}$, for $H$ with respect to $\{V_j\}_{j \in J}$, is defined by

$$Sf = \sum_{j \in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in H,$$

where $\Lambda_j^*$ is the adjoint operator of $\Lambda_j$.

It is easy to check that $S$ is a bounded, invertible and self-adjoint operator and

$$AI \leq S \leq BI, \quad \frac{1}{B} I \leq S^{-1} \leq \frac{1}{A} I.$$

Writing $\tilde{\Lambda}_j = \Lambda_j S^{-1}$, then for any $f \in H$ we have

$$f = \sum_{j \in J} \Lambda_j^* \Lambda_j f = \sum_{j \in J} \tilde{\Lambda}_j^* \Lambda_j f.$$
It is prove that the sequence $\Lambda_j S^{-1}$ is also a g-frame (called canonical dual g-frame) for $H$ with respect to $\{V_j\}_{j \in J}$. For more details we refer to [5].

In this work we present two iterative methods in order to approximate the solution of the operator equation

$$Lu = f,$$

where $L : H \to H$ is bounded invertible and symmetric operator on a separable Hilbert space $H$. In [1, 3, 4] you can see some developments of numerical methods for solving this problem by using frames.

2 Using g-frames in Richardson iterative method

In this chapter by using Richardson iterative method and g-frames, we wish to solve the operator equation (1). First of all we give and exact solution by using a g-frame.

Theorem 2.1. Let $L : H \to H$ be a bounded and invertible operator and $\{\Lambda_j\}_{j \in J}$ be a g-frame for $H$. Then $\{\Lambda_j L\}_{j \in J}$ is also a g-frame for $H$.

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:

Write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, define for $k \geq 0$,

$$u_{k+1} = (I - L)u_k + f.$$

Since $Lu - f = 0$,

$$u_{k+1} - u = (I - L)u_k + f - u - (f - Lu) = (I - L)u_k - u + Lu = (I - L)(u_k - u).$$

Hence

$$\|u_{k+1} - u\|_H \leq \|I - L\|_{H \to H} \|u_k - u\|_H,$$

so that (2) converges if

$$\|I - L\|_{H \to H} < 1.$$

It is sometimes possible to precondition (1) by multiplying both sides by a matrix $B$,

$$BLu = Bf,$$

so that convergence of iterative methods is improved. This is very effective technique for solving differential equations, integral equations, and related problems. The following theorem designs an iterative method based on Richardson iterative method and knowledge of g-frames.

Theorem 2.2. Let $\{\Lambda_j\}_{j \in J}$ be a g-frame with g-frame operator $S$, and $A$ and $B$ be the bounds of the g-frame $\{\Lambda_j L\}_{j \in J}$.

Put $u_0 = 0$ and for $k \geq 1$, $u_k = u_{k-1} + \frac{2}{A+B}LS(f - Lu_{k-1})$, Then

$$\|u - u_k\| \leq (\frac{B - A}{B + A})^k \|u\|.$$

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3 Chebyshev method by using g-frames

Before introducing our next method, we wish to state without proof some basic facts about Chebyshev polynomials. These polynomial are defined by

\[ c_n(x) = \begin{cases} 
\cos(n \cos^{-1}(x)), & |x| \leq 1 \\
cosh(n \cosh^{-1}(x)) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n} \right), & |x| \geq 1 
\end{cases} \]

and satisfy the recurrence relation

\[ c_0(x) = 1, \quad c_1(x) = x, \quad c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x), \quad \forall n \geq 2. \]

The following lemma holds.

**Lemma 3.1.** Given \( a \leq b \leq 1 \), set \( P_n(x) = \frac{c_n(\frac{2x-a-b}{b-a})}{c_n(\frac{2x-a-b}{b-a})} \), then

\[ \max_{a \leq x \leq b} |P_n(x)| \leq \max_{a \leq x \leq b} |Q_n(x)| \]

for all polynomial \( Q_n \) of degree \( n \) satisfying \( Q_n(1) = 1 \). Furthermore

\[ \max_{a \leq x \leq b} |P_n(x)| = \frac{1}{c_n(\frac{2-a-b}{b-a})}. \]

Now let \( h_n = \sum_{k=1}^{n} a_{nk}u_k \) such that \( \sum_{k=1}^{n} a_{nk} = 1 \), where \( u_k \) is the approximated solutions induced by the iterative method represented in the Theorem 2.2.

The condition \( \sum_{k=1}^{n} a_{nk} = 1 \) guaranteed if \( u_1 = u_2 = ... = u_n = u \), then \( h_n = \sum_{k=1}^{n} a_{nk}u_k = u \sum_{k=1}^{n} a_{nk} = u \).

In this case, by Theorem 2.2

\[ u - h_n = \sum_{k=1}^{n} a_{nk}u - \sum_{k=1}^{n} a_{nk}u_k = \sum_{k=1}^{n} a_{nk}(u - u_k) = \sum_{k=1}^{n} a_{nk}(I - \frac{2}{A + B}LSL)^k(u - u_0). \]

Writing \( R = I - \frac{2}{A + B}LSL \) and \( Q_n(x) = \sum_{k=1}^{n} a_{nk}x^k \), we obtain

\[ u - h_n = \sum_{k=1}^{n} a_{nk}R^k(u - u_0) = Q_n(R)(u - u_0), \]

that means the error is a polynomial in \( R \) applied to the initial error \( u - u_0 \).

Also we note that the spectrum of \( R \) is obtained in \([-\rho, \rho] \) where \( \rho = \frac{B-A}{B+A} \), and since \( LSL \) is a positive definite operator, the spectral theorem yields

\[ ||u - h_n|| \leq ||Q_n(R)|| ||u - u_0|| \leq \max_{|x| \leq \rho} |Q_n(x)| ||u - u_0||. \]

In order to minimize this error we try to find

\[ \min_{Q_n(1)=1} \max_{|x| \leq \rho} |Q_n(x)|, \tag{3} \]

where the min is taken over all polynomials of degree less than or equal to \( n \), with \( Q_n(1) = \sum_{k=1}^{n} a_{nk} = 1 \). By Lemma 3.1 the answer can be given in terms of the Chebyshev
First we note that, replacing $a = -B - A/B + A$ and $b = B - A/B + A$ in Lemma 3.1 gives

$$P_n(x) = \frac{c_n(2x + B - A/B + A)}{c_n(2 + B - A/B + A)} = \frac{c_n(z)}{c_n(\frac{z}{\rho})}.$$  

This polynomials solve (3). Now, based on the above argument we can organize the following algorithm in order to induce an approximated solution to the equation (1). Let $\{\Lambda_j\}_{j \in J}$ be a g-frame for $H$ with frame operator $S$ and let $A$ and $B$ be the bounds of the g-frame $\{\Lambda_jL\}_{j \in J}$.

**Algorithm** $[A, B, \epsilon] \rightarrow u_{\epsilon}$

(i) put $\rho = \frac{B - A}{B + A}$, $\sigma = \sqrt{B + \sqrt{A}}$, $\beta_1 = 2$, $n = 1$

(ii) while $\frac{2\sigma^n}{1+\sigma^n} \frac{\|f\|}{m} > \epsilon$

1. $n = n + 1$
2. $\beta_n = (1 - \frac{\rho^2}{4} \beta_{n-1})^{-1}$
3. $h_n = \frac{2}{\rho} \beta_n (h_{n-1} + \frac{2}{B + A} LSF (f - Lh_{n-1})) + (1 - \beta_n)h_{n-2}$

(iii) $u_{\epsilon} := h_n$.

The following theorem verifies the convergence of this algorithm.

**Theorem 3.2.** If $u$ is the exact solution of the equation (1) then, the approximated solution $h_n$ satisfies $\|u - h_n\| \leq \frac{2\sigma^n}{1+\sigma^n} \frac{\|f\|}{m}$. Consequently the output $u_{\epsilon}$ in the Algorithm $[A, B, \epsilon]$ satisfies

$$\|u - u_{\epsilon}\| < \epsilon.$$  

**Remark 3.3.** It is obvious that for every $n > 1$, $\frac{2\sigma^n}{1+\sigma^n} \leq \rho^n$. Therefore this algorithm present an iterative method that is convergence is faster than the Richardson iterative method that is presented in Theorem 2.2.

**References**


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