An effective algorithm for reachability of linear control problems in minimum time

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Abstract. In this paper minimum time for a class of linear control problems is discussed. It is assumed that $u(t)$, control function, is continuous in minimum time linear control problem (MTLCP). A new problem in calculus of variations form which is equivalent to MTLCP is defined. In this new problem, it is discussed how polynomials with unknown constant coefficients can be substituted with the state and control functions. After this substitution, a sequence of minimum time nonlinear problems (MTNP) which its solution converge to the solution of original problem (MTLCP), is obtained. For solving the sequence of MTNPs an algorithm which give polynomials as the approximations of optimal state and control functions and the approximation of minimum time, is proposed. In this method, the error of approximations and the convergence of algorithm are considered. Finally, the efficiency of this approach is confirmed by some numerical examples.

2. problem statement

Consider the following MTLCP:

$$\min \ t_f$$
$$s.t.\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ \quad U^- \le u(t) \le U^+ \\ \quad x(t_0) = x_0 \ \ x(t_f) = x_1, \end{cases} \tag{1}$$

Where $t \in [t_0, t_f]$, t_0 is known positive real number and t_f is unknown real number, $U^- = (a_1,..., a_m)$ and $U^+ = (b_1,..., b_m)$ are given m-dimensional vectors of real numbers, $x(t) = (x_1(t),..., x_n(t))^t$ is a n-dimensional vector of the state functions such that $x_i(t) \in C^1(A_i)$ and A_i is closed interval in real number, $u(t) = (u_1(t),..., u_m(t))^t$ is a m-dimensional vector of the control functions such that $u_i(t) \in C([a_i, b_i])$, $x_0 = (x_{01},..., x_{0n})$ and $x_1 = (x_{11},..., x_{1n})$ are given initial and final state in $\Re^n$, respectively, and $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrixes, respectively, where its entries are continuous functions on $I = [t_0, t_f]$.

Consider the system (1), we define the error functional $E(\dot{x}(t), x(t), u(t), t)$ as follows:

$E(\dot{x}(t), x(t), u(t), t) = \|\dot{x}(t) - (A(t)x(t) + B(t)u(t))\|_2^2$, $\forall t \in I$, Where $\|.\|_2$ is Euclidian norm in $\Re^n$.

Now the following problem is defined:

$$\min \ t_f$$
$$s.t.\begin{cases} \int_{t_0}^{t_f} E(\dot{x}(t), x(t), u(t), t)\,dt = 0 \\ \int_{t_0}^{t_f} \left\| u(t) - U^+ + |u(t) - U^+| \right\|_2^2 dt = 0 \\ \int_{t_0}^{t_f} \left\| u(t) - U^- - |u(t) - U^-| \right\|_2^2 dt = 0 \\ x(t_0) = x_0 \qquad x(t_f) = x_1, \end{cases} \tag{2}$$

Theorem 2. The system (1) and (2) are equivalent.

Now, let $p_r(t) = (p_{r,1}(t),..., p_{r,n}(t))^t$, $q_s(t) = (q_{s,1}(t),..., q_{s,m}(t))^t$, Where $p_{r,i}(t), i = 1,..., n,$ and $q_{s,j}(t), j = 1,..., m,$ are the polynomials of r and s degrees ($r, s = 1,2,...$) with unknown constant coefficients as the approximations of $x_i(t)$ and $u_j(t)$, respectively. Then by setting $p_r(t)$ and $p_s(t)$ instead of $x(t)$ and $u(t)$ in system (2) respectively, we get a sequence of MTNPs as

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follows:

$$\min\ t_f$$

$$s.t.\begin{cases}\int_{t_0}^{t_f} E(\dot{p}_r(t), p_r(t), q_s(t), t)\,dt = 0 \\ \int_{t_0}^{t_f} \left\| q_s(t) - U^+ + \left| q_s(t) - U^+ \right| \right\|_2^2 dt = 0 \\ \int_{t_0}^{t_f} \left\| q_s(t) - U^- - \left| q_s(t) - U^- \right| \right\|_2^2 dt = 0 \\ p_r(t_0) = x_0 \qquad p_r(t_f) = x_1 \end{cases} \qquad r,s = 1,2,..., \tag{3}$$

We suppose Q be the set of t that for any t there are continuous functions $x(t)$ and $u(t)$ such that the system (2) is feasible, and $Q_{r,s}$ be the set of t that for any t there are $p_r(t)$ and $q_s(t)$ such that system (3) is feasible. Also we suppose $Q \neq \phi$, $Q_{r,s} \neq \phi$.

Theorem 3. If $t^* = \min_Q t$ and $t_{r,s} = \min_{Q_{r,s}} t$ then $\lim_{r,s\to\infty} t_{r,s} = t^*$.

3. A practical method for the solution of MTLPs

We partition interval $[t_0, t_f]$ to n parts such that the length of any partition is equal to a constant number such as h ($h = (t_f - t_0)/n$). Now, we assume original problem is system (3). Thus by using a numerical integration method such as trapezoidal rule, The system (3) is converted to a MTNP and we may acquire its solution by many packages such as Matlab, Lingo and…. Finally, an algorithm is proposed for obtaining minimum-time in linear control problems. In this algorithm, the error of between two derived consecutive times and the approximations of trapezoidal rule, is considered.

Example. Consider linear time-variant reachable problem as follows:

$$\dot{x} = x + t^2 u$$
$$x(0) = -1 \quad x(t_f) = 0 \qquad |u| \le 1$$

We choose $\varepsilon_t = 10^{-5}$, $\varepsilon_R = 10^{-5}$, then after the few iteration of algorithm, we get $u(t) = 1$, $x(t) = -1 - 0.986009t - 0.5822917t^2 + 0.3057606t^3 - 0.05330168t^4 + 0.03422093t^5$ and $t^* = 2.674105$, where the exact solution of problem is $u(t) = 1$ and $x(t) = e^t - 2 - 2t - t^2$ with $t^* = 2.6741$.

4. Conclusions

We are considering the above method on minimum time nonlinear control problems and so far, good result have got. This method has some advantages. For example, in this method, the number of variables is lower than the direct methods and there are no costate variables as exist in indirect methods.

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