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INTEGRAL CLOSURES AND HOMOLOGICAL DIMENSIONS

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Abstract Throughout this talk, all rings considered will be commutative Noetherian and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by I. Let K be a non-zero finitely generated module over R. The concept of *weakly G_K-perfect ideal* was introduced by Golod in [5]. He showed that, this new ideal has some nice properties. For instance, he proved that for a weakly G_K-perfect ideal I of R, K is a canonical module for R if and only if $\mathrm{Ext}^s_R(R/I,K)$ is a canonical module for R/I, where $s = \mathrm{grade}_K I$.

The concept of integral closure of an ideal I of R was introduced by Northcott and Rees. Let I_a denote the integral closure of I in R, i.e., I_a is the ideal of R consisting of all elements $x \in R$ which satisfy an equation $x^n + r_1x^{n-1} + \cdots + r_n = 0$, where $r_i \in I^i$, $i = 1,2,\ldots,n$. On the other hand, Golod in [5] defined the important notion of a suitable module K and used it to define G_K-dimension, as a refinement of projective dimension, for finitely generated modules. The G_K-dimension of a finitely generated R-module M is the length of the shortest resolution of M by so-called totally K-reflexive modules, motivated by Gerko in [4] has extended this notion to complexes. Also, in [4] Gerko introduced the interesting concept of Cohen-Macaulay dimension which characterizes the Cohen-Macaulay rings. The aim of this paper is to study weakly G_K-perfect ideals and the structure of associated primes to the integral closure of ideals.

We shall say that an ideal I of R is *K-proper* if $K/IK \neq 0$, and when this is the case, we define the *$K-$height* of I (written $\mathrm{ht}_K I$) to be

$$\inf\{\mathrm{ht}_K \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}\, K \cap V(I)\}(= \inf\{\mathrm{ht}_K \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_R K/IK\}).$$

For any K-proper ideal I of R, we denote by $\mathrm{grade}_K I$ the maximum length of all K-sequences contained in I. A well known result of Rees shows that $\mathrm{grade}_K I$ is the least integer i such that $\mathrm{Ext}^i_R(R/I,K) \neq 0$. We say that I is *weakly G_K-perfect* if $\mathrm{Ext}^i_R(R/I,K) = 0$ for $i \neq \mathrm{grade}_K I$.

In the first section we give a characterization for canonical modules. Namely, we show that if $(R,\mathfrak{m})$ is a local ring, and K is a non-zero finitely generated suitable R-module, then $\mathfrak{m}$ is G_K-perfect if and only if K is a canonical module for R.

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The concept of a suitable (or semidualizing) module was introduced by Golod [5]. An R-module K is said to be *suitable (semidualizing)* if the natural homomorphism $R \longrightarrow \mathrm{Hom}_R(K,K)$ is an isomorphism, and $\mathrm{Ext}^i_R(K,K)=0$ for all $i>0$. Every suitable module is a PG-module, which was introduced by Foxby [2] as a common generalization of a projective module and a Gorenstein module. We recall that an R-module G is called a PG-module if $\mathrm{Hom}_R(G,G)$ is projective and $\mathrm{Ext}^i_R(G,G)=0$ for all $i>0$.

The notion of Gorenstein dimension (G-dimension) was introduced by Auslander [1] and was deeply studied by him and Bridger. Foxby defined the concept of G-dimension with respect to a PG-module, and Golod [5] used it to define the concept of G-dimension with respect to a suitable module K (short: G_K-dimension) as follows: For any R-module X let X^* (or, when necessary, X^*_K) denote $\mathrm{Hom}_R(X,K)$. We say that X is *K-reflexive* if the natural homomorphism $X \longrightarrow X^{**}$ is an isomorphism. Let $G_K(R)$ be the class of all R-modules such that X is K-reflexive and $\mathrm{Ext}^i_R(X,K)=0=\mathrm{Ext}^i_R(X^*,K)$ for all $i>0$. If there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to X \to 0$$

of R-modules such that $X_i \in G_K(R)$ for each $0 \le i \le n-1$ and $X_n \ne 0$, then we say that X has *G_K-dimension at most n*, and write $G_K\text{-}\mathrm{dim}_R X \le n$. If such integer n does not exist, then we say that X has *infinite G_K-dimension*, and write $G_K\text{-}\mathrm{dim}_R X = \infty$. If an R-module X has G_K-dimension at most n but does not have G_K-dimension at most $n-1$, then we say that X has *G_K-dimension n*, and write $G_K\text{-}\mathrm{dim}_R X = n$. We say that M is *G_K-perfect* if $G_K\text{-}\mathrm{dim}_R X = \mathrm{grade}\, X$; such modules was introduced under the name *quasi-perfect* by Foxby [3].
In the second section we study the structure of associated primes to the integral closure of ideals which have finite homological dimension. In fact we show that, if K is a finitely generated suitable R-module, I is a K-proper integrally closed ideal of R such that $G_K\text{-}\mathrm{dim}_R I < \infty$ and K satisfies Serre's condition (S_1) or $\mathrm{grade}_K I > 0$, then $K_\mathfrak{p}$ is a canonical $R_\mathfrak{p}$- module for every $\mathfrak{p} \in \mathrm{Ass}_R R/I$.

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