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BANASCHEWSKI'S THEOREM FOR S-POSETS: REGULAR INJECTIVITY AND COMPLETENESS

M. MAHMOUDI AND H. RASOULI

Faculty of Mathematical Sciences
Shahid Beheshti University
19839, Evin, Tehran, Iran `hrasouli5@yahoo.com`

Abstract. In this talk, the notion of injectivity in the category **Pos-S** of S-posets, for a pomonoid S, is discussed. First we see that, although there is no non-trivial injective S-poset with respect to monomorphisms, **Pos-S** has enough (regular) injectives with respect to regular monomorphisms (sub S-posets). Then, recalling Banaschewski's theorem which states that regular injectivity of posets with respect to order-embeddings and completeness are equivalent, we study it for S-posets and get some homological classification of pomonoids and pogroups. Among other things, we also see that regular injective S-posets are exactly the retracts of cofree S-posets over complete posets.

1. Introduction

Banaschewski [1] proves that complete posets are exactly injective posets relative to extremal monomorphisms (order-embeddings), and Sikorski [3] shows the same result for injective Boolean algebras. The main objective of this paper is to study the counterpart of Banaschewski's Theorem for S-posets; posets on which the actions of the pomonoid S on them preserve the order.

Definition 1.1. A *left poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid left ideal ($SI \subseteq I$) and a down set ($a \leq b, b \in I$ imply $a \in I$).

Definition 1.2. A pomonoid S which has no proper non-empty left poideal is said to be *left simple*.

Definition 1.3. By a *complete S-poset*, we mean an S-poset which is merely complete as a poset.

Lemma 1.4. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$. Also, let $\mathcal{M}, \mathcal{M}'$ be certain subclasses of $\mathcal{C}, \mathcal{D}$, respectively. If for all $f \in \mathcal{M}$, $Ff \in \mathcal{M}'$, then for any $\mathcal{M}'$-injective object $D \in \mathcal{D}$, GD is an $\mathcal{M}$-injective object of $\mathcal{C}$.*

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2. Injectivity and regular injectivity in **Pos-S**

Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. Then we have:

Lemma 2.1. **Pos** *has no non-trivial injective object.*

Theorem 2.2. **Pos-S** *has no non-trivial injective object.*

Lemma 2.3. *Every non-trivial (non-singleton) regular injective S-poset A is bounded by two zero elements.*

Corollary 2.4. *Let the identity of the pomonoid S be its top element. If S regarded as an S-poset is regular injective then $S = \{1\}$.*

Theorem 2.5 (**Pos**-S has enough regular injectives)**.** *Each S-poset can be regularly embedded into a regular injective S-poset.*

Theorem 2.6. *An S-poset is regular injective if and only if every regular embedding $A \to B$ has a left inverse.*

Theorem 2.7. *An S-poset A is regular injective if and only if it is a retract of a cofree S-poset over a complete poset.*

3. Regular injectivity and completeness

Proposition 3.1. *Every regular injective S-poset is complete.*

Since a regular injective S-poset must have two zero elements (Lemma 2.3) and since this is not necessarily the case for complete ones, the converse of the above proposition is not true.

Theorem 3.2. *An S-poset A is complete if and only if $A^{(S)}$ is a regular injective S-poset.*

Theorem 3.3. *Let S be a pomonoid. If all complete S-posets are regular injective, then S is left simple.*

Theorem 3.4. *Let S be a commutative pomonoid or a pomonoid whose identity is the top element. Then all complete posets are regular injective as S-posets with trivial actions.*

Corollary 3.5. *If the chain* **2** *is a regular injective S-poset then its action is trivial. The converse is true if S is a commutative pomonoid or a pomonoid whose identity is the top element.*

Lemma 3.6. *Let S be a pogroup and A be a complete S-poset. Then, for any $X \subseteq A$ and $s \in S$, $(\bigvee X)s = \bigvee Xs$.*

Theorem 3.7. *Let S be a pogroup. Then an S-poset is complete if and only if it is regular injective.*

Proposition 3.8. *Let S be a pomonoid whose identity is a maximal (respectively, the top) element. Then all complete S-posets are regular injective if and only if S is a pogroup (respectively, $S = \{1\}$).*

Remark 3.9. We know that if S is a group, any S-act is injective if and only if it has a zero element. But, this fact does not hold for regular injectivity in **Pos-S**. For example, taking $S = \{1\}$, and A to be any incomplete S-poset, then all elements of A are zero while A is not regular injective.

Theorem 3.10. *Let S be a pogroup. Then all complete posets are regular injective as S-posets with trivial actions.*

Remark 3.11. The converse of the above theorem is not generally true. In fact, if S is a pomonoid and all complete posets are regular injective S-posets with trivial actions, then S is not necessarily a pogroup. For example, consider the pomonoid $S = (\mathbb{N}^\infty, min, \leq)$. Since S is commutative, by Theorem 3.4, all complete S-posets with trivial actions are regular injective, whereas S is not a pogroup.

References

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