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# RELATIONS BETWEEN BLOCKING SETS AND $\mathfrak{C}_n$-GROUPS

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ABSTRACT. A blocking set in $PG(n,q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n,q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. A cover $\mathcal{C}$ for a group $G$ is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free $n$-cover for $G$ and in this case we say that $G$ is a $\mathfrak{C}_n$-group.

In this paper we give relations between blocking sets and $\mathfrak{C}_n$-groups.

## 1. Introduction

Let $G$ be a group. A set $\mathcal{C}$ of proper subgroups of $G$ is called a cover for $G$ if its set-theoretic union is equal to $G$. If the size of $\mathcal{C}$ is $n$, we call $\mathcal{C}$ an $n$-cover for the group $G$. A cover $\mathcal{C}$ for a group $G$ is called irredundant if no proper subset of $\mathcal{C}$ is a cover for $G$. A cover $\mathcal{C}$ for a group $G$ is called core-free if the intersection $D = \bigcap_{M \in \mathcal{C}} M$ of $\mathcal{C}$ is core-free in $G$, i.e. $D_G = \bigcap_{g \in G} g^{-1}Dg$ is the trivial subgroup of $G$. A cover $\mathcal{C}$ for a group $G$ is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of $G$. A cover $\mathcal{C}$ for a group $G$ is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free $n$-cover for $G$ and in this case we say that $G$ is a $\mathfrak{C}_n$-group.

A blocking set in $PG(n,q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n,q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. For a blocking set $B$ in $PG(n,q)$ we denote by $d(B)$ the least positive integer $d$ such that $B$ is contained in a $d$-dimensional subspace of $PG(n,q)$. Thus $d(B)$ is equal to the (projective) dimension of the subspace $\langle B \rangle$ in $PG(n,q)$.

Nontrivial minimal blocking sets in $PG(2,p)$ of size $\frac{3(p+1)}{2}$ exist for all odd primes $p$. Indeed, an example is given by the projective triangle: the set consisting of the points $(0,1,-s^2)$, $(1,-s^2,0)$, $(-s^2,0,1)$ with $s \in \mathbb{F}_p$.

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For further studies in the topic of blocking sets see Chapter 13 of the second edition of Hirschfeld's book [4] and also see [3] and for further studies in the topic of covering groups by subgroups see [1] and [2].

In this paper we give relations between blocking sets and $\mathfrak{C}_n$-groups.

## 2. Main results

As we mentioned in the last section, by a blocking set in $PG(n, q)$, we mean a blocking set with respect to hyperplanes in $PG(n, q)$.

Now we give some notations and definitions as needed in the sequel. We denote the product of $n$ copies of $\mathbb{F}_q$ by $(\mathbb{F}_q)^n$. We note that $(\mathbb{F}_q)^n$ is a vector space of dimension $n$ over $\mathbb{F}_q$. If $b = (b_1, \ldots, b_n) \in (\mathbb{F}_q)^n$, we denote by $M_b$ the set of elements $x = (x_1, \ldots, x_n) \in (\mathbb{F}_q)^n$ such that $b \cdot x = \sum_{i=1}^{n} b_i x_i$ is equal to zero. Note that if $0 \neq b$, then $M_b$ is an $(n-1)$-dimensional subspace of the vector space $(\mathbb{F}_q)^n$ and every $(n-1)$-dimensional subspace of $(\mathbb{F}_q)^n$ equals to $M_b$ for some non-zero $b \in (\mathbb{F}_q)^n$. Since for every $0 \neq \lambda \in \mathbb{F}_q$, $M_b = M_{\lambda b}$, $M_{\mathfrak{p}}$ is well-defined for every point $\mathfrak{p}$ of $PG(n-1, q)$, and $M_{\mathfrak{p}}$ may be considered as a hyperplane in $PG(n-1, q)$. We now give some results which clarify the relations between non-trivial minimal blocking sets of size $n$ and $\mathfrak{C}_n$-covers for groups.

**Proposition 2.1.** *Let $B$ be a set of points in $PG(n, q)$. Then $B$ is a blocking set in $PG(n, q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

**Proposition 2.2.** *Let $B$ be a set of points in $PG(n, q)$. Then $B$ is a minimal blocking set in $PG(n, q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is an irredundant $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

**Remark 2.3.** *Note that if $q$ is prime, then the cover $\mathcal{C}$ in the statements of Propositions 2.1 and 2.2 is a maximal cover for $(\mathbb{F}_q)^{n+1}$.*

**Remark 2.4.** *It is easy to see that a (minimal) blocking set $B$ with $d(B) = d$ in $PG(n, q)$ can be obtained from a (minimal) blocking set in $PG(d, q)$. So if we adopt an induction process on $n$ to find all minimal blocking sets $B$ in $PG(n, q)$, we must find only all those minimal blocking sets with $d(B) = n$.*

**Proposition 2.5.** *Let $B$ be a set of points in $PG(n, q)$. Then $B$ is a blocking set with $d(B) = n$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a core-free $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

**Theorem 2.6.** *Let $p$ be a prime number and $n$ be a positive integer. Then a finite $p$-group $G$ admits a $\mathfrak{C}_{n+1}$-cover if and only if $G \cong (C_p)^{m+1}$ for some positive integer $m$ such that $PG(m, p)$ has a minimal blocking set $B$ with $d(B) = m$ and $|B| = n + 1$.*

**Proof.** Let $G$ be a finite $p$-group admitting a $\mathfrak{C}_{n+1}$-cover. Then $G$ has a maximal irredundant core-free $(n+1)$-cover, $\mathcal{C} = \{M_i \mid i = 1, \ldots, n+1\}$ say. Since the Frattini subgroup $\Phi(G)$ of $G$ is contained in $M_i$ for every $i \in \{1, \ldots, n+1\}$, $\Phi(G) \leq D_G = 1$, where $D$ is the intersection of the cover $\mathcal{C}$. Hence $\Phi(G) = 1$ and so $G$ is isomorphic to $(C_p)^{m+1}$ for some positive integer $m$. Now Propositions 2.2 and 2.5 and Remark 2.3 complete the proof.

## References

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