G-FRAMES AND STABILITY OF G-FRAMES IN HILBERT SPACES

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ABSTRACT. In 2006 Wenchang Sun introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. We present a version of the Paley-Wiener Theorem for g-frames which is in spirit close to results for frames, due to Ole Christensen.

1. Introduction

There are some generalizations of frames, the most recent of these generalizations is g-frame. This is an extension of frames that include all of the previous extensions of frames.

Through this paper, $H$ and $K$ are Hilbert spaces and $\{H_i : i \in I\}$ is a sequence of Hilbert spaces, where $I$ is a subset of $\mathbb{Z}$. The space $\mathcal{L}(H, H_i)$ is the collection of all bounded linear operators from $H$ to $H_i$.

Note that for any sequence $\{H_i : i \in I\}$, we can assume that there exists a Hilbert space $\mathcal{K}$ such that for all $i \in I$, $H_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = \bigoplus_{i \in I} H_i$).

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Definition 1.1. We call a sequence \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a generalized frame, or simply a g-frame for \( H \) with respect to \( \{ H_i : i \in I \} \), if there exist two positive constants \( A \) and \( B \) such that
\[
A \| f \|^2 \leq \sum_{i \in I} \| \Lambda_i f \|^2 \leq B \| f \|^2, \quad f \in H.
\]
We call \( A \) and \( B \) the lower and upper g-frame bounds, respectively.

We say also a g-frame for \( H \) with respect to \( K \) whenever \( H_i = K \) for each \( i \in I \).

We call \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a g-frame sequence, if it is a g-frame for \( \text{span} \{ \Lambda_i^*(H_i) \}_{i \in I} \).

Definition 1.2. Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \). Then the synthesis operator for \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is the operator
\[
T: \left( \bigoplus_{i \in I} H_i \right) \ell_2 \rightarrow H
\]
defined by
\[
T(\{ f_i \}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i).
\]
We call the adjoint \( T^* \) of the synthesis operator is the analysis operator.

The operator \( S = TT^* \) is called the g-frame operator.

It is easy to show that
\[
f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f,
\]
for every \( f \in H \).

2. Perturbation of g-frames

Theorem 2.1. Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \) with bounds \( A, B \) and \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a sequence of operators such that for any finite subset \( J \subseteq I \) and for each \( f \in H \),
\[
\left\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \right\| \leq \lambda \left( \sum_{i \in J} \Lambda_i^* \Lambda_i f \right) + \mu \left( \sum_{i \in J} \Theta_i^* \Theta_i f \right) + \gamma \left( \sum_{i \in J} \| \Lambda_i f \|^2 \right)^{\frac{1}{2}},
\]
where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a g-frame for $H$ with frame bounds

$$A \frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \quad \text{and} \quad B \frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu}.$$  

Corollary 2.2. Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a g-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of operators. If there exists a constant $0 < R < A$ such that

$$\sum_{i \in I} \|\Lambda_i^* f - \Theta_i^* f\| \leq R\|f\|$$

for all $f \in H$, then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a g-frame with g-frame bounds $A - R$ and $\min\left\{1 + R\sqrt{\frac{B}{A}}, R + B\right\}$.

Theorem 2.3. Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a g-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of operators such that for every $J \subseteq I$ with $|J| < +\infty$,

$$\left\|\sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i)\right\|$$

$$\leq \lambda \left\|\sum_{i \in J} \Lambda_i^* f_i\right\| + \mu \left\|\sum_{i \in J} \Theta_i^* f_i\right\| + \gamma \left(\sum_{i \in J} \|f_i\|^2\right)^{1/2}, \quad (f_i \in H_i)$$

where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a g-frame for $H$ with g-frame bounds

$$A \left(1 - (\lambda + \frac{\gamma}{\sqrt{A}})^2\right)^{1/2} \quad \text{and} \quad B \left(1 + \lambda + \frac{\gamma}{\sqrt{B}}\right)^2.$$  

Proposition 2.4. Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a g-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of operators. If there exists an $R$ with $0 < R < A$ such that

$$\sum_{i \in I} \|\Lambda_i f - \Theta_i f\|^2 \leq R\|f\|^2$$

for all $f \in H$. Then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a g-frame for $H$ with bounds $(\sqrt{A} - \sqrt{R})^2$ and $(\sqrt{B} + \sqrt{R})^2$.

Theorem 2.5. Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a g-frame for $H$ with respect to $\{H_i : i \in I\}$, and $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of
operators. If
\[ K \left( \sum_{i \in I} \bigoplus H_i \right)_{\ell^2} \rightarrow H, \quad K(f_i)_{i \in I} = \sum_{i \in I} (\Lambda_i^* - \Theta_i^*) f_i \]
is a well-defined and compact operator, then \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a g-frame sequence.

**Corollary 2.6.** Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \). Let \( J \) be a finite subset of \( I \) such that for each \( j \in J \), \( \dim H_j < \infty \). Then \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \setminus J \} \) is a g-frame sequence.

**Theorem 2.7.** Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \) and let \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a family of operators. If
\[ K : H \rightarrow H, \quad K f = \sum_{i \in I} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \]
is a well-defined and compact operator, then \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a g-frame sequence.

**References**


