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A CHARACTERIZATION OF ARVESON SPECTRUM ON LOCALLY COMPACT HYPERGROUPS

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Abstract. Spectral subspaces were introduced by R. Godement, and a systematic study of spectral subspaces and their applications to dynamic systems was presented by W. Arveson. We have already defined the Arveson spectrum and spectral subspaces for hypergroups and studied their basic properties. Here we give a characterization of Arveson spectrum on locally compact commutative hypergroups.

1. Introduction

Hypergroups were introduced in a series of papers by R. I. Jewett, C. F. Dunkl, and R. Spector in 70 's. They are in fact extensions of topological groups. Roughly speaking, a hypergroup is a locally compact space which necessarily does not have an algebraic structure but it has enough structure so that a convolution on the space of finite regular Borel measures can be defined. Therefore, the extension of Fourier analysis on hypergroups is made with more difficulties and sometimes with different proofs to that of groups. Examples include locally compact groups, double-coset hypergroups, polynomial hypergroups, etc.

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First we recall the definition of a hypergroup. The main references are [1] and [3].

Definition 1.1. *Let K be a locally compact Hausdorff space. The space K is called a commtative hypergroup if there exists a binary mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $M^+(K)$ satisfying the following conditions,*

(1) *The mapping $(\delta_x, \delta_y) \mapsto \delta_x * \delta_y$ extends to a bilinear associative operator $*$ from $M(K) \times M(K)$ into $M(K)$ such that*
$$\int_K f d(\mu * \nu) = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\nu(y)$$
for all continuous functions f on K vanishing at infinity.

(2) *For each $x, y \in K$ the measure $\delta_x * \delta_y$ is a probability measure with compact support.*

(3) *The mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M^+(K) \times M^+(K)$ into $M^+(K)$; the topology on $M^+(K)$ being the cone topology.*

(4) *There exists an $e \in K$ such that $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$ for all $x \in K$.*

(5) *There exists a homeomorphic involution $x \mapsto x^-$ from K onto K such that, for all $x, y \in K$, we have $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ where for $\mu \in M(K)$, μ^- is defined by $\int_K f(t) d\mu^-(t) = \int_K f(t^-) d\mu(t)$, and also, $e \in supp(\delta_x * \delta_y)$ if and only if $y = x^-$, where $supp(\delta_x * \delta_y)$ is the support of the measure $\delta_x * \delta_y$.*

(6) *The mapping $(x, y) \mapsto supp(\delta_x * \delta_y)$ from $K \times K$ into the space $\boldsymbol{C}(K)$ of compact subsets of K is continuous, where $\boldsymbol{C}(K)$ is given the topology whose subbasis is given by all $\boldsymbol{C}_U(V) = \{A \in \boldsymbol{C}(K) : A \cap U \neq \emptyset \text{ and } A \subseteq V\}$, where U , V are open subsets of K.*

(7) *For each $x, y \in K$, $\delta_x * \delta_y = \delta_y * \delta_x$.*

Throughout this paper K is a locally compact commutative hypergroup.

Definition 1.2. *Let M be a W^*-algebra with predual space M_*. Let $\sigma : M(K) \to B_\sigma(M)$ be a norm-decreasing algebra-homomorphism. For each $t \in K$ we denote $\sigma_t = \sigma(\delta_t)$. The mapping σ is called a representation if:*

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(1) *for each* $t \in K$, $\sigma_t : M \to M$ *is an* $*-$*automorphism;*
(2) *for every* $x \in M$ *and* $\rho \in M_*$, *the function* $t \mapsto < \sigma_t(x), \rho >$ *is continuous;*
(3) $\sigma_e = I_M$, *where* e *is the identity of* K *and* I_M *is the identity mapping on* M.

For each $\mu \in M(K)$ we have

$$< \sigma(\mu)(x), \rho >= \int_K < \sigma_t(x), \rho > \, d\mu(t),$$

where $x \in M$ and $\rho \in M_*$ [5].

Definition 1.3. *Let* $\sigma : M(K) \to B_\sigma(M)$ *be a representation.*

(i) *The Arveson spectrum of* σ *is defined by* $sp\sigma := hull(\{f \in L^1(K) : \sigma(f) = 0\})$. *Trivially* $sp\sigma = \bigcap\{\hat{f}^{-1}(0) : f \in L^1(K) \, and \, \sigma(f) = 0\}$.
Since for each $f \in L^1(K)$, $\hat{f} \in C_0(\hat{K})$, $sp\sigma$ *is a closed subset of* $\hat{K}$. *Also for each* $\rho \in M_*$ *and* $f \in L^1(K)$,

$$< \sigma(f)(1_M), \rho >= \int_K < \sigma_t(1_M), \rho > \, f(t)\, dm(t) =< \hat{f}(1)1_M, \rho > .$$

Therefore for each $f \in L^1(K)$, $\sigma(f)(1_M) = \hat{f}(1)1_M$, *and so if* $\sigma(f) = 0$ *then* $\hat{f}(1) = 0$. *In other words* $1 \in sp\sigma$.
(ii) *Let* $x \in M$. *We define* $sp_\sigma(x) := hull(\{f \in L^1(K) : \sigma(f)(x) = 0\})$. *Then* $sp_\sigma(x)$ *is a closed subset of* $\hat{K}$.
(iii) *Let* E *be a closed subset of* $\hat{K}$. *We define the associated spectral subspace by* $M(\sigma, E) := \{x \in M : sp_\sigma(x) \subseteq E\}$.

Spectral subspaces were introduced by R. Godement, which may be viewed as an attempt to extend the Stone theorem. A systematic study of spectral subspaces and their applications to dynamic systems was presented by W. Arveson. In [5] we have defined the Arveson spectrum and spectral subspaces for a commutative strong hypergroup K and we also assume that $X_b(K) = \hat{K}$. We studied their basic properties on hypergroups and presented some examples of hypergroups of this type. Here we give a characterization of Arveson spectrum on locally compact hypergroups. The proof of Lemma 2.1 is technical and completely different from the group case because its proof in the case of groups is based on $(fg)_x = f_x g_x$, while this relation is not true in general for

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hypergroups. This lemma helps us to extend a significant result from A. Connes [2], to hypergroups.

2. Main Results

Lemma 2.1. *Let* $\epsilon > 0$, ξ *be in the center of* $\hat{K}$, $f \in L^1(K)$ *and* $\hat{f}(\xi) = 0$. *Then there exists a function* $k \in L^1(K)$ *such that* $\|f * k\|_1 < \epsilon$ *and* $\hat{k} \equiv 1$ *on a neighborhood of* ξ.

Lemma 2.2. *Let* $\xi \in Z(\hat{K})$ *and* $\epsilon > 0$. *For every compact set* $E \subseteq K$ *there exists a compact neighborhood* V *of* ξ *in* $\hat{K}$ *such that for each* $s \in E$ *and* $x \in M(\sigma, V)$,

$$\|\sigma_s(x) - \overline{\xi(s)}x\| < \epsilon \, \|x\|.$$

Summing up, we have the following characterization of spσ.

Theorem 2.3. *Let* ξ *be in the center of* $\hat{K}$. *The following properties are equivalent.*

(i) $\xi \in sp\sigma$.

(ii) *For each closed neighborhood* V *of* ξ *in* $\hat{K}$, $M(\sigma, V) \neq \{0\}$.

(iii) *There is a net* (x_ι) *in* M *such that for each* ι, $\|x_\iota\| = 1$ *and*

$$\lim_{\iota} \|\sigma_s(x_\iota) - \xi(s)x_\iota\| = 0,$$

uniformly on compacta.

(iv) *For every* $f \in L^1(K)$, $|\hat{f}(\xi)| \leq \|\sigma(f)\|$.

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