The 18^{th} *Seminar on Mathematical Analysis and its Applications*
26-27 Farvardin, 1388 (15-16 April, 2009) pp 190-193
Tarbiat Moallem University

CERTAIN DENSE SUBSPACES OF EXTENDED LITTLE LIPSCHITZ ALGEBRAS

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Abstract. Let (X, d) be a compact metric space. In 1987 Bade, Curtis and Dales determined certain dense subspaces of $lip(X, \alpha)$, when $0 < \alpha < 1$. Let K be a nonempty closed subset of X. In this paper we introduce new classes of Lipschitz algebras $Lip(X, K, \alpha)$ and $lip(X, K, \alpha)$, consisting of those continuous complex-valued functions f on X such that $f|_K \in Lip(K, \alpha)$, $lip(K, \alpha)$, respectively, and determine certain dense subspaces of $lip(X, K, \alpha)$, when $0 < \alpha < 1$.

1. Introduction

Let (X, d) be a compact metric space and $\alpha > 0$. The algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\Big\{\frac{|f(z) - f(w)|}{(d(z, w))^\alpha} : z \neq w \ , z, w \in X\Big\} < \infty,$$

is denoted by $Lip(X, \alpha)$. If we set $\|f\|_\alpha = \|f\|_X + p_\alpha(f) (f \in Lip(X, \alpha))$, then $(Lip(X, \alpha), \|.\|_\alpha)$ is a commutative Banach algebra, which is called the Lipchitz algebra of order α on X. The set of all complex-valued functions f on X for which

2000 Mathematics Subject Classification: 46J10, 46J15.
keywords and phrases: Second dual, Lipschitz algebra, extended Lipschitz algebras.

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$$\frac{|f(z)-f(w)|}{(d(z,w))^{\alpha}} \longrightarrow 0 \ as \ d(z,w) \longrightarrow 0\,,$$

is a closed subalgebra of the Lipschitz algebra $(Lip(X,\alpha), \|.\|_{\alpha})$, which is denoted by $lip(X,\alpha)$ and called little Lipschitz algebra of order α on X. The Lipschitz algebras $Lip(X,\alpha)$ for $\alpha \in (0,1]$ and $lip(X,\alpha)$ for $\alpha \in (0,1)$, are Banach function algebras on X. Extensive study of Lipschitrz algebras started with Sherbert [3], [4]. T. G. Honary and S. Moradi introduced a new class of analytic Lipschitz algebras in the plane in 2007 [2]. In this paper we introduce more general classes of Lipschitz algebras for every metric space (X,d) and determine some properties of them.

Definition 1.1. *Let K be a nonempty closed subset of the compact metric space (X,d) and let α be a positive number. For each $f \in C(X)$, set*

$$p_{\alpha,K}(f) := p_{\alpha}(f|_K) = \sup\Big\{\frac{|f(z)-f(w)|}{(d(z,w))^{\alpha}} : z \neq w, z, w \in K\Big\}.$$

We denote by $Lip(X,K,\alpha)$ the set of all $f \in C(X)$ for which $f|_K \in Lip(K,\alpha)$ and by $lip(X,K,\alpha)$ the set of all $f \in C(X)$ for which $f|_K \in lip(K,\alpha)$.

It is easy to see that $Lip(X,K,\alpha)$ $(lip(X,K,\alpha))$ under the norm

$$\|f\|_{\alpha,K} := \|f\|_X + p_{\alpha,K}(f)$$

is a Banach function algebra when $0 < \alpha \leq 1$ $(0 < \alpha < 1)$.

Note. In this paper (X,d) is a compact metric space and K is a compact subset of X.

2. Certain Dense Linear Subspaces of $lip(X,K,\alpha)$

Bade, Curtis and Dales in [1; Theorem 3.6] determined certain dense linear subalgebras of $lip(X,\alpha)$ in 1987 as follows:

Theorem 2.1. *Let P be a linear subspace of $lip(X,\alpha)$. Suppose that there is a constant C such that for each finite subset E of X and for each $f \in lip(X,\alpha)$, there exists $g \in P$ with $g|_E = f|_E$ and with $\|g\|_{\alpha} \leq C\|f\|_{\alpha}$. Then P is dense in $lip(X,\alpha)$.*

We now generalize the above theorem for $lip(X,K,\alpha)$ as follows:

Theorem 2.2. *Let P be a linear subspace of $lip(X, K, \alpha)$ which satisfy the following conditions:*
(a) If $h \in C(X)$ with $h|_K = 0$, then $h \in \overline{P}$, where $\overline{P}$ is the closure of P in $lip(X, K, \alpha)$.
(b) There is a constant C such that for each finite subset E of K and for each $f \in lip(X, K, \alpha)$, there exists $g \in P$ such that $g|_E = f|_E$ and $\|g\|_{\alpha,K} \leq C\|f\|_{\alpha,K}$.
(i) Suppose $P_K := \{g \in C(K) : \text{there is } f \in P \text{ such that } g = f|_K\}$. Then, P_K is a dense linear subspace of $lip(K, \alpha)$.
(ii) P is dense in $lip(X, K, \alpha)$.

proof. (i) It is clear that P_K is a linear subspace of $lip(K, \alpha)$. Let E be a finite subset of K and let $f \in lip(K, \alpha)$. By Tietze's extension theorem, there exists $F \in C(X)$ such that $F|_K = f$ and $\|F\|_X = \|f\|_K$. Clearly $F \in lip(X, K, \alpha)$. By (b), there exists $G \in P$ such that $G|_E = F|_E$ and $\|G\|_{\alpha,K} \leq C\|F\|_{\alpha,K}$. Set $g = G|_K$. Then $g \in P_K$, $g|_E = f|_E$ and

$$\begin{aligned} \|g\|_\alpha &= \|g\|_K + p_\alpha(g) \leq \|G\|_X + p_\alpha(g) \\ &= \|G\|_X + p_{\alpha,K}(G) = \|G\|_{\alpha,K} \\ &\leq C\|F\|_{\alpha,K} = C(\|F\|_X + p_{\alpha,K}(F)) \\ &= C(\|f\|_K + p_\alpha(f)) = C\|f\|_\alpha. \end{aligned}$$

Thus P_K is dense in $lip(K, \alpha)$, by Theorem 2.1.
(ii) It is enough to show that if $\Phi \in (lip(X, K, \alpha), \|.\|_{\alpha,K})^*$ with $\Phi(f) = 0$ for all $f \in P$, then $\Phi(f) = 0$ for all $f \in lip(X, K, \alpha)$. □

We now show that $Lip(X, K, 1)$ is dense in $lip(X, K, \alpha)$, by applying the above theorem. For this purpose, we need the following lemma:

Lemma 2.3 (1, Lemma 2.3)**.** *For each finite subset E of X and each $h \in Lip(X, \alpha)$, there exists $f \in Lip(X, 1)$ such that $f|_E = h|_E$ and $\|f\|_\alpha \leq 2\|h\|_\alpha$.*

We now generalize the above lemma.

Lemma 2.4. *For each finite subset E of X and each $h \in Lip(X, K, \alpha)$, there exists $f \in Lip(X, K, 1)$ such that $f|_E = h|_E$ and $\|f\|_{\alpha,K} \leq 3\|h\|_{\alpha,K}$.*

proof. Let E be a finite subset of X and let $h \in Lip(X, K, \alpha)$. Set $g = h|_K$. Then $g \in Lip(K, \alpha)$. By Lemma 2.3, there exists $g_0 \in$

$Lip(K,1)$ with $g_0|_{E\cap K} = g|_{E\cap K}$ and $\|g_0\|_\alpha \leq 2\|g\|_\alpha$. We now define the function $g_1 : E \cup K \longrightarrow \mathbb{C}$ by

$$g_1(z) = \begin{cases} g_0(z) & z \in K \\ h(z) & z \in E\backslash K. \end{cases}$$

Clearly, $E \cup K$ is a compact subset of X and $g_1 \in C(E \cup K)$. By the Tietze's extension theorem, there exists $f \in C(X)$ such that $f|_{E\cup K} = g_1$ and $\|f\|_X = \|g_1\|_{E\cup K}$. It follows that $f \in Lip(X,K,1)$ and $f|_E = h|_E$. Furthermore,

$$\begin{aligned} \|f\|_{\alpha,K} &= \|f\|_X + p_{\alpha,K}(f) = \|g_1\|_{E\cup K} + p_\alpha(g_0) \leq \|g_1\|_K + \|g_1\|_{E\backslash K} + p_\alpha(g_0) \\ &= \|g_0\|_K + \|h\|_{E\backslash K} + p_\alpha(g_0) = \|g_0\|_\alpha + \|h\|_{E\backslash K} \leq 2\|g\|_\alpha + \|h\|_{E\backslash K} \\ &= 2(\|g\|_K + p_\alpha(g)) + \|h\|_{E\backslash K} = 2\|g\|_K + \|h\|_{E\backslash K} + 2p_{\alpha,K}(h) \\ &\leq 2\|h\|_X + \|h\|_{E\backslash K} + 2p_{\alpha,K}(h) \leq 3\|h\|_{\alpha,K}. \end{aligned}$$

□

Theorem 2.5. *$Lip(X,K,1)$ is dense in $lip(X,K,\alpha)$.*

proof. Set $P = Lip(X,K,1)$. It is known that P is a linear subspace of $lip(X,K,\alpha)$. Also if $h \in C(X)$ and $h|_K = 0$, then $h \in P$.

Let E be a finite subset of K and $f \in lip(X,K,\alpha)$. Then there exists $g \in P$ with $g|_E = f|_E$ and $\|g\|_{\alpha,K} \leq 3\|f\|_{\alpha,K}$, by the above lemma. Therefore, P is dense in $lip(X,K,\alpha)$, by Theorem 2.2. □

Corollary 2.6. [1, Corollary 3.7] *$Lip(X,1)$ is dense in $lip(X,\alpha)$.*

proof. It is enough to take $K = X$ in Theorem 2.5. □

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