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TRACE CLASS OPERATOR INEQUALITIES

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Abstract. This paper is an exposition of some formulations of the fundamental real-number inequalities in trace class operators.

1. Introduction

First we introduce some notation and conventions. Suppose that H is a complex, separable Hilbert space and $B(H)$ is the algebra of bounded linear operators acting on H. For an operator $x \in B(H)$, $|x|$ shall denote $(x^*x)^{\frac{1}{2}}$, the positive root of x^*x. Also we denote by $R[x]$ the projection of H to closed range $\overline{\text{ran} x}$. If $z \in B(H)$ is compact, then nonnegative real numbers $s_k(z)$, for every $k \in \mathrm{Z}^+$, denote the k-th singular value of z, namely

$$s_k(z) = \lambda_k(|z|),$$

where

$$\lambda_k(|z|) = \text{Min}\,\{\text{Max}\,\{\langle |z|\xi, \xi\rangle \,:\, \xi \in M^{\perp},\ \|\xi\| = 1\},\ M \subset H,\ \dim M = k-1\}.$$

Trace-class and Hilbert Schmidt operators are defined via the sequence of singular values. An operator $x \in B(H)$ is of trace class if $\{s_k(x)\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$, and x is a Hilbert Schmidt operator if $\{s_k(x)\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Thus, the trace norm $\|\cdot\|_1$ and Hilbert Schmidt norm $\|\cdot\|_2$ on

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the ideals of trace class operators and Hilbert Schmidt operators, respectively, are:

$$\|x\|_1 = \sum_{k=1}^{\infty} s_k(x)$$

and

$$\|x\|_2 = \left(\sum_{k=1}^{\infty} s_k(x)^2\right)^{1/2}.$$

The trace of trace class operator x is defined to be

$$\operatorname{tr}(x) = \sum_{k=1}^{\infty} \langle s\phi_k, \phi_k \rangle,$$

where $\{\phi_k\}_{k\in\mathbb{N}}$ is any orthonormal basis of H. It is well known that the trace is independent of the choice of orthonormal basis and that $\operatorname{tr}(xg) = \operatorname{tr}(gx)$ if $x \in B(H)$ is of trace class operator and $g \in B(H)$ is arbitrary.

If a is a positive compact operator, then the list $\{\lambda_k(x)\}_{k\in\mathbb{N}}$ accounts for all nonzero eigenvalues of a. Hence,

$$\|a\|_1 = \operatorname{tr}(a) = \sum_{k=1}^{\infty} \lambda_k(x).$$

Finally, we use some basic facts from the theory of memorization, for which our main reference is the book of Bhatia[3].

2. Inequalities

We begin by mentioning the trivial fact that if $p,\ q \in \mathbb{R}^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $p > 1$ and $q > 1$. Thus, if $a, b \in B(H)$ are positive trace class operators, then both a^p and b^q are positive trace class operators. The following young inequality version for singular values of compact operators was shown in [4].

Theorem 2.1. *If a and b are two positive compact operators and $p,\ q \in \mathbb{R}^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for ever $k \in \mathbb{N}$,*

$$\lambda_k(|ab|) \le \lambda_k\left(\frac{a^p}{p} + \frac{b^q}{q}\right).$$

For Young's inequality in compact operators, there are no known characterisations of cases of equality. However, some partial results concerning cases of equality have been obtained recently for trace-class and Hilbert–Schmidt operators.

Hirzallah and Kittaneh [5] proved that if $a, b \in \mathcal{K}(H)$ are positive Hilbert–Schmidt operators such that

$$\|ab\|_2 = \|p^{-1}a^p + q^{-1}b^q\|_2 ,$$

then $b^q = a^p$. This result can be used, via the methods of [2], to characterise in the operator Young inequality under the assumption that x and y are Hilbert–Schmidt operators. Argerami and Farenick [2] considered the case of trace-class operators, proving the following results.

Theorem 2.2. *Let $x, y \in B(H)$ be trace-class operators and suppose that $u \in B(H)$ is a partial isometry. Then the following statements are equivalent:*

(1) $u|xy^*|u^* = p^{-1}|x|^p + q^{-1}|y|^q$;

(2) $|y|^q = |x|^p$, $R[u^*] \geq R[y]$, *and* $uy = v^*(yu)v$,

where $y = v|y|$ is the polar decomposition of y.

Theorem 2.3. *If $a, b \in B(H)$ are positive trace-class operators and if $u \in B(H)$ is a partial isometry, then the following conditions are equivalent:*

(1) $\|u|ab|u^*\|_1 = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_1$;

(2) $\|u|ab|u^*\|_2 = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_2$;

(3) $|a|^p = |b|^q$ *and* $R[u^*] \geq R[b]$.

Another fundamental inequality between complex numbers is the triangle inequality. As a consequence result of Akemann, Anderson, and Pedersen[1], we have the following proposition.

Proposition 2.4. *If x and y are trace class operators in $B(H)$, then there are partial isometries $u,\ v \in B(H)$ such that*

$$|x + y| \leq u^*|x|u + v^*|y|v.$$

If $\dim(H) < \infty$, then u and v are unitaries and equality holds if and only if $x = u|x|$ and $y = u|y|$. We do not know under what conditions the equality holds in trace class operators.

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