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# NORM ESTIMATES OF THE PRE-SCHWARZIAN DERIVATIVE FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS INVOLVING THE JUNG-KIM-SRIVASTAVA OPERATOR

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Abstract. By making use of subordination between analytic functions and the Jung-Kim-Srivastava operator, we introduce certain subclass of univalent functions. Then we estimate the norm of the pre-Schwarzian derivative of this class.

## 1. Introduction

Let $\mathcal{H}$ denote the class of all analytic functions the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. Also let $\mathcal{S}$ denote the class of all univalent functions in $\mathcal{A}$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the familiar classes of functions in $\mathcal{A}$ that are starlike (with respect to origin) and convex, respectively. A function $f \in \mathcal{S}$ is called starlike of order $\alpha$, $0 \leq \alpha < 1$, if and only if

$$Re\ \left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ (z \in D). \tag{1}$$

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We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order $\alpha$ in $D$. A function $f \in \mathcal{S}$ is called convex of order $\alpha$, $(0 \leq \alpha < 1)$, if and only if

$$Re\ \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ (z \in D). \tag{2}$$

We denote by $\mathcal{K}(\alpha)$ the class of all function in $\mathcal{S}$ which are convex of order $\alpha$ in $D$. It follows from (1) and (2) that

$$f(z) \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha). \tag{3}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [4]. For analytic functions $g$ and $h$ in $D$, $g$ is said to be subordinate to $h$ if there exists an analytic function $\omega$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = h(\omega(z))$ for $z \in D$. The subordination will be denoted by $g \prec h$, or, conventionally, $g(z) \prec h(z)$. We now introduce the terminology needed below. Let $\mathcal{M}$ be the class of non-vanishing analytic functions $\phi$ in $D$ with the normalized condition $\phi(0) = 1$. Following Ma and Minda [3], we define the subclasses $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ of $\mathcal{A}$ as the sets of functions $f \in \mathcal{A}$ of the forms

$$\frac{zf'(z)}{f(z)} \prec \phi(z),$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),$$

respectively, for each $\phi \in \mathcal{M}$. By definition, it is obvious that $f(z) \in \mathcal{K}(\phi)$ if and only if $zf'(z) \in \mathcal{S}^*(\phi)$. We note that $\mathcal{S}^*(\phi) \subset \mathcal{S}^*(\psi)$ and $\mathcal{K}(\phi) \subset \mathcal{K}(\psi)$ for $\phi \prec \psi$. In 1993, Jung et al. [1] introduced the following integral operator:

$$I^\sigma f(z) := \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(log\frac{z}{t}\right)^{\sigma-1} dt\ (\sigma > 0), \tag{4}$$

for the functions $f(z) \in \mathcal{A}$. Let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, which are analytic and convex and satisfy the condition $Re\ (p(z)) > 0$, $(z \in D)$.

**Definition 1.1.** *A function $f \in \mathcal{A}^+$ is said to be in the class $\mathcal{S}^\sigma(\eta; h)$ if it satisfies the following differential subordination:*

$$\frac{1}{1-\eta}\left(\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} - \eta\right) \prec h(z)\ (z \in D;\ 0 \leq \eta < 1;\quad h \in \mathcal{P}). \tag{5}$$

*where* $\mathcal{A}^+$ *denotes the class of functions of the form* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $(a_n > 0)$, *which are analytic in* $D$. *For simplicity, we write*

$$\mathcal{S}^\sigma\left(\eta; \frac{1+Az}{1+Bz}\right) =: \mathcal{S}^\sigma(\eta; A, B)\ (-1 \le B < A \le 1).$$

Kim and Merkes considered the nonlinear integral transform $J_\alpha$ defined by

$$J_\alpha[f](z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha d\zeta \tag{6}$$

for complex numbers $\alpha$ and for functions $f$ in the class

$$\mathcal{ZF} := \{f \in \mathcal{A} : f(z) \neq 0 \text{ forall } z \text{ with } 0 < |z| < 1\},$$

and showed that $J_\alpha(\mathcal{S}) = \{J_\alpha[f] : f \in \mathcal{S}\} \subset \mathcal{S}$ when $|\alpha| < \frac{1}{4}$.

Let $f : D \longrightarrow \mathbb{C}$ be analytic and locally univalent. The pre-Schwarzian derivative $T_f$ of $f$ is defined by

$$T_f(z) = \frac{f''z)}{f'(z)}.$$

We also define the norm of $T_f$ by

$$\|T_f\| = \sup_{z\in D} |T_f(z)|(1-|z|^2).$$

It is also known that $\|T_f\| \le 6$ for $f(z) \in \mathcal{S}$ and that $\|T_f\| \le 4$ for $f(z) \in \mathcal{K}$ [2], and, conversely, for $f \in \mathcal{A}$, $\|T_f\| \le 1$ implies $f \in \mathcal{S}$ (Becker's theorem). The aim of this paper is to give norm estimates of the pre-Schwarzian derivative for the class $\mathcal{S}^\sigma(\eta; h)$.

## 2. Main Results

**Theorem 2.1.** *If* $f \in \mathcal{S}^\sigma(\eta; h)$, *then* $\|T_F\| \le 4$.

**Proof.** Let $f \in \mathcal{S}^\sigma(\eta; h)$. Since $\frac{1}{1-\eta}\left(\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} - \eta\right) \prec h(z)$, there exists a Schwarz function $\omega$, which is analytic in $D$ with

$$\omega(0) = 0 \textit{ and } |\omega(z)| < 1\ (z \in D)$$

such that

$$\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} = (1-\eta)h(\omega(z)) + \eta.$$

If we set $g(z) = (1-\eta)h(z) + \eta$, then we see that $g$ is analytic in $D$ and $g(0) = 1$. Therefore, $I^\sigma f(z) \in \mathcal{S}^*(g)$, and so $F := J[I^\sigma] \in \mathcal{K}(g)$. Now by [3, Theorem 2.1], we easily obtain $f' \prec \mathcal{K}_g'$, where

$$\mathcal{K}_g(z) = \int_0^z \left(\exp \int_0^\zeta \frac{g(t)-1}{t} dt\right) d\zeta.$$

In general, for $f, g \in \mathcal{A}$, the condition $f' \prec g'$ implies the inequality $\|T_f\| \le \|T_g\|$. Hence, we obtain

$$\|T_F\| \le \|T_{\mathcal{K}_g}\|. \tag{7}$$

Since $Re\ h > 0$ and $0 \le \eta < 1$, we get

$$Re\ g = Re\ ((1-\eta)h + \eta) = (1-\eta) Re\ h + \eta > 0.$$

This implies that $Re\ \left(1 + \frac{z\mathcal{K}_g''}{\mathcal{K}_g'}\right) = Re\ g > 0$, or equivalently $\mathcal{K}_g \in \mathcal{K}$. Hence, $\|T_{\mathcal{K}_g}\| \le 4$. Now, by (7) we obtain

$$\|T_F\| \le 4,$$

and the proof is now complete. □

**Theorem 2.2.** *Let* $0 \le \eta < 1$ *and* $-1 \le A < B \le 1$. *If* $f \in \mathcal{S}^\sigma(\eta; A, B)$, *then*

$$\|T_F\| \le \frac{2(1-\eta)(A-B)}{1+\sqrt{1-B^2}}.$$

**Corollary 2.3.** *Let* $0 \le \eta < 1$ *and* $-1 \le A < B \le 1$. *If* $f \in \mathcal{S}^\sigma(\eta; A, B)$, *and* $\frac{2(1-\eta)(A-B)}{1+\sqrt{1-B^2}} < 1$, *then* $F \in \mathcal{S}$ *and* $I^\sigma f \in \mathcal{K}$.

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