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PRESERVERS OF QUASI-COMMUTATIVITY ON $\mathbf{M}_n$ OR $\mathbf{H}_n$

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ABSTRACT. Let $\mathbf{M}_n$ and $\mathbf{H}_n$ be the algebra of all $n \times n$ complex matrices and the space of all $n \times n$ Hermitian matrices, respectively. For $A, B \in \mathbf{M}_n$, it is said that A quasi-commute with B and we denote by $A \sim_q B$ if AB and BA are linearly dependent and either $AB = 0 = BA$ or they are both nonzero. This is equivalent to the existence of a nonzero $\lambda = \lambda_{A,B} \in \mathbb{C}$ such that $AB = \lambda BA$. The relation of quasi-commutativity has applications in quantum mechanics. Recently, there has been quite some activity in classifying the linear maps, which preserve this relation. In this note we present the general form of linear preserver of $\sim_q$ on $\mathbf{M}_n$ and strong preserver (not necessarily linear) of $\sim_q$ on $\mathbf{H}_n$.

1. Introduction

For some given relations among matrices, some linear preserver problems deal with the characterizations of linear maps on matrix spaces which preserve these relations. Let $\mathcal{A}$ be a linear space of matrices and $\sim$ be a relation on $\mathcal{A}$. We say that a linear operator $T : \mathcal{A} \longrightarrow \mathcal{A}$

(a) *preserves* $\sim$, if $T(X) \sim T(Y)$ whenever $X \sim Y$.

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(b) *strongly preserves* $\sim$, if $T(X) \sim T(Y)$ if and only if $X \sim Y$.
These problems represent one of the most extensively investigated research areas in matrix theory over the past decades. We refer the reader to [1] and [3], to see the structure of some kinds of linear preservers.

2. Quasi-commutativity on $\mathbf{M}_n$

The aim of this Section is to consider some linear preservers problem for the relation of quasi-commutativity on $\mathbf{M}_n$. The following statements give some information about linear preservers of $\sim_q$ on $\mathbf{M}_n$. For every $A \in \mathbf{M}_n$, put

$$\mathcal{C}(A) = \{B : AB = BA\} \text{ and } A^{\#} = \{B : A \sim_q B\}.$$

It is easy to see that $\mathcal{C}(A)$ is a subspace of $\mathbf{M}_n$ but in general $A^{\#}$ is not a subspace of $\mathbf{M}_n$. Molnar in [4] found a necessary and sufficient condition for which $A^{\#}$ is a subspace of $\mathbf{M}_n$.

Theorem 2.1. *Let $0 \neq A \in \mathbf{M}_n$. Then $A^{\#}$ forms a linear subspace of $\mathbf{M}_n$ if and only if there exist invertible matrices $S \in \mathbf{M}_n$ and $J \in \mathbf{M}_r$ $(1 \leq r \leq n)$ such that J has exactly one eigenvalue and $A = S^{-1}(J \oplus 0)S$. Moreover, in this case we have $B \in A^{\#}$ if and only if $AB = BA$.*

For every strong linear preserver T of $\sim_q$ on $\mathbf{M}_n$, if $T(A) = 0$ then $T(A)T(B) = T(B)T(A)$ and hence $A \sim_q B$ for all $B \in \mathbf{M}_n$. Therefore, $A \in \text{span}\{I\}$. So we obtain the following proposition:

Proposition 2.2. *If $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ is a strong linear preserver of $\sim_q$, then T is bijective or Ker(T) =span$\{I\}$.*

The following example is a strong linear preserver of $\sim_q$, which is not bijective.

Examples 2.3. *Let $T : \mathbf{M}_2 \longrightarrow \mathbf{M}_2$ be defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}$. It is easy to see that T strongly preserves $\sim_q$ and Ker(T) =Span$\{I\}$.*

In [4], the author used Theorem 2.1 to establish a characterization of bijective strong linear preservers of $\sim_q$ on $\mathbf{M}_n$, for $n \geq 2$.

Theorem 2.4. *Let $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ be a bijective linear map. Then T strongly preserves $\sim_q$ if and only if there exist a nonzero scalar $\alpha \in \mathbb{C}$ and a nonsingular matrix $S \in \mathbf{M}_n$ such that*

$$T(A) = \alpha S^{-1}AS \ (A \in \mathbf{M}_n) \text{ or } T(A) = \alpha S^{-1}A^tS \ (A \in \mathbf{M}_n)$$

In [5], Radjavi and Semrel extended the last theorem and found the structure of linear preservers of $\sim_q$ on $\mathbf{M}_n$, for $n \geq 2$.

Theorem 2.5. *For $n \geq 3$, let $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ be a linear operator. Then T preserves $\sim_q$, if and only if one of the following three conditions holds:*
(1) *the range of T is commutative.*
(2) *the range of T is anti-commutative.*
(3) *there exist a nonzero scalar $\alpha \in \mathbb{C}$ and a nonsingular matrix $S \in \mathbf{M}_n$ such that*

$$T(A) = \alpha S^{-1}AS \ (A \in \mathbf{M}_n) \ \textit{or} \ T(A) = \alpha S^{-1}A^tS \ (A \in \mathbf{M}_n).$$

In the next theorem f is one of the following linear functionals:

$$f(A) = trace(A) \text{ or } f(A) = \frac{1}{2}trace(A)$$

Theorem 2.6. *Let $T : \mathbf{M}_2 \longrightarrow \mathbf{M}_2$ be a linear map. Then T preserves $\sim_q$, if and only if one of the following holds:*
(1) *the range of T is commutative.*
(2) *there exist a nonzero scalar $\alpha \in \mathbb{C}$ and a nonsingular matrix $S \in \mathbf{M}_n$ such that*

$$T(A) = \alpha[S^{-1}AS - f(A)I] \ (A \in \mathbf{M}_n) \ \textit{or} \ T(A) = \alpha[S^{-1}A^tS - f(A)I] \ (A \in \mathbf{M}_n)$$

3. Quasi-commutativity on $\mathbf{H}_n$

Since in quantum mechanics the space of Hermitian operators plays a distinguished role, it is natural to consider the preserver problem also on $\mathbf{H}_n$. In case of Hermitian matrices, we have the following special phenomenon.

Proposition 3.1. *Let $A, B \in \mathbf{H}_n$. Then $A \sim_q B$ if and only if they commute ($AB = BA$) or anti-commute ($AB = -BA$). Furthermore,*

$$A^{\#} \cap \mathbf{H}_n = \{C : AC = CA\} \cup \{C : AC = -CA\}$$

Let $\mathcal{R}$ be the following relation on $\mathbf{H}_n$:

$$\mathcal{R}(A, B) \text{ if and only if } A^{\#} \cap \mathbf{H}_n = B^{\#} \cap \mathbf{H}_n,$$

for all $A, B \in \mathbf{H}_n$. It is easy to verify that $\mathcal{R}$ is an equivalence relation on $\mathbf{H}_n$. For every $A \in \mathbf{H}_n$, the equivalence class of A with respect to $\mathcal{R}$ is denoted by $[A]$. In [2], Dolinar and Kuzma characterized those maps preserving $\sim_q$ on $\mathbf{H}_n$, for $n \geq 3$.

Theorem 3.2. *Let $\varphi : \mathbf{H}_n \longrightarrow \mathbf{H}_n$ be a bijective strong preserver of $\sim_q$. Then there exists a unitary matrix $U \in \mathbf{M}_n$ such that*

$$\varphi(A) \in [U^*AU] \quad (A \in \mathbf{H}_n) \text{ or } \varphi(A) \in [U^*A^tU] \ (A \in \mathbf{H}_n)$$

For $n \geq 2$, a characterization of bijective strong linear preserver of $\sim_q$ on $\mathbf{H}_n$ has been obtained in [4] .

Theorem 3.3. *Let $T : \mathbf{H}_n \longrightarrow \mathbf{H}_n$ be a bijective linear map which strongly preserves $\sim_q$. Then there exist a nonzero scalar $\alpha \in \mathbb{R}$ and a unitary matrix $U \in \mathbf{M}_n$ such that*

$$T(A) = \alpha U^*AU \ (A \in \mathbf{H}_n) \text{ or } T(A) = \alpha U^*A^tU \ (A \in \mathbf{H}_n) \ (1)$$

Examples 3.4. *The restriction of the linear operator T in Example* 2.3 *to $\mathbf{H}_n$, is a strong linear preserver of $\sim_q$ on $\mathbf{H}_n$, which is not of the form* (1).

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