مقایسه روش‌های پایدارسازی مستقیم و تکراری در پایدارسازی مسئله انتقال به سمت پایین تعیین زئوئید

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چکیده

مسئله انتقال به سمت پایین میدان گرانی زمین از سطح زمین به سطح پوشی مرجع مقایسه از این واقعیت ناشی می‌شود که در مسئله مقادیر مرزی، تعیین زئوئید بدون استفاده از فرمول استوکس به‌دلیل توانستن انفعالی زمین روی سطح بخشی مرجع متغیر به طرق خودکار شیب زمین می‌باشد. در صورت اینجاست مسئله انتقال به سمت پایین میدان گرانش زمین از طریق انتگرال ابیل-پویسون و مشابهات آن دارای دو صورت پذیرفته و یک مسئله به‌طور داده شده است. برای اینصدای انتگرال ابیل-پویسون، رویهای پایدارسازی مستقیم و تکراری به‌طور انتقال به سمت پایین شده‌اند. مشاهدات از نوع شتاب گرانی تفاضلی مقایسه و روش تکراری ART به‌عنوان بهترین روش برای پایدارسازی معرفی شده است.

واژه‌های کلیدی: مسأله زئوئید، مسئله انتقال به سمت پایین، انتگرال ابیل-پویسون، رویهای پایدارسازی

A Comparison of direct and indirect regularization methods for downward continuation problem of geoid computations without applying Stokes formula

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Abstract

The problem of downward continuation of the gravity field from the Earth’s surface to the reference ellipsoid arises from the fact that the solution to the boundary value problem for geoid determination without applying Stokes formula is sought in terms of the disturbing potential \( h W(X) \) on the ellipsoid but the disturbing gravity observations \( \delta \Gamma(X) \) are only available on the Earth’s surface. Downward continuation is achieved via Abel-Poisson integral and its derivatives. Using discrete observations, the Abel-Poisson integral has to be transformed into a summation form:

\[
b = Ax, \quad b \in \mathbb{R}^m, \quad x \in \mathbb{R}^n\]

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Where the matrix $A$ is the design matrix and $b$ stands for the disturbing gravity observations vector. The downward continuation problem is an inverse problem. Inverse problems are ill-posed, like any ill-posed problem it must be regularized. The objective of this paper is the comparison between direct and iterative methods for solving downward continuation of the gravity field from the Earth’s surface to the reference ellipsoid for geoid determination without applying Stokes formula.

Direct regularization methods are methods where the solution is directly derived. In this contribution truncated method, standard Tikhonov method and generalized Tikhonov method using discretized norms at Sobolov subspaces $W^1_2(a,b)$, $W^2_2(a,b)$ and Sobolov semi norms $\|L^1\|$ and $\|L^2\|$ are implemented. Based on SVD, in truncated methods, the solution can be obtained as:

$$ x_{\text{Reg}} = \sum_{i=1}^{r_{\lambda}} \frac{\langle u_i, b \rangle}{\sigma_i} v_i $$

Where $u_i$ and $v_i$ are the right and the left singular vectors, respectively. $r_{\lambda}$ is rank of matrix $A_{\lambda}$ that is a L2 norm approximation for matrix $A$. In the case of TGSVD the solution is obtained as

$$ x_{\text{Reg}} = \sum_{i=p_{\lambda}+1}^{n} \frac{\langle u_i, b \rangle}{\sigma_i} x_i + \sum_{i=p_{\lambda}+1}^{n} \langle u_i, b \rangle x_i $$

In standard Tikhonov method, the minimizing function can be written as:

$$ F_{\text{Tikhonov}}(x;\lambda) = \|Ax - b\|_2^2 + \lambda \|x\|_2^2 $$

In this method, filter coefficients and solution become:

$$ f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda} $$

$$ x_{\text{Reg}} = \sum_{i=1}^{n} f_i \frac{u_i^\top b}{\sigma_i} v_i $$

In standard Tikhonov method, the matrix $L$ was $I_m$. In generalized Tikhonov method, we select the matrix $L$ as follows

$$ L = \begin{pmatrix} \alpha_1 L^s \\
\vdots \\
\alpha_s L^1 \\
\alpha_s L^0 \end{pmatrix} $$

Where the $\{L^i\}, i = 1, 2, \ldots, s$ is obtained from discretization of derivative operators up to order $s$ and coefficients $\{\alpha_i\}, i = 1, \ldots, s$ are weight coefficients.

In contrast to direct methods, in iterative methods, normal equations are solved via construction of a sequence of the solutions that converge to the pseudo-inverse solution of the equations. In this contribution classical iterative method, Landweber-Fridman method, Tikhonov iterative method, Algebraic Reconstruction Technique (ART), conjugate
gradient method and LSQR method are implemented.

Classical iterative methods are based on construction of sequences of solutions \( \{x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots\} \). For the matrix equation \( Ax = b \), the following relationship holds between solution \( x^{(k)} \) and solution \( x^{(k+1)} \):

\[
x^{(k+1)} = \left( I - Q^{-1} A^T A \right) x^{(k)} + A^T b
\]

In Landweber-Fridman method the matrix \( Q^{-1} \) is equal to diagonal matrix \( \omega I \). Ergo, in this method, iterative relation between the solutions is defined as:

\[
x^{(k+1)} = (I - \omega A^T A) x^{(k)} + \omega A^T b
\]

In Tikhonov iterative method, iterative relation between the solutions is defined as:

\[
x^{(k+1)} = \left( A^T A + \lambda I \right)^{-1} (A^T b + \lambda x^{(k)})
\]

The idea of Algebraic Reconstruction Technique iteration to solve the matrix equation \( Ax = b \) is to partition the system row wise, either into single rows or into blocks of rows. Each of these rows defines a hyper plane of dimension \( n - 1 \). The idea of the ART iteration is to project the current approximate solution successively onto each one of these hyper planes. It turns out that such a procedure converges to the solution of the system.

A best known method for solving large scale equations system is conjugate gradient. Conjugate gradient is a type of Krylov subspace method. Conjugate gradient method is suitable for positive definite operators.

In LSQR method, solution vector is defined as follows:

\[
x^{(k)}_{[LSQR]} = \beta_k V_k B_k e^{(k)}_1
\]

Where right vectors \( V_k \) can be found in Hansen (1998) and \( B_k \) is a bidiagonal matrix with \( \alpha_k \) and \( \beta_k \) on the main diagonals and \( e^{(k+1)}_1 = (1, 0, \ldots, 0)^T \).

For comparison of different regularization methods and the selection of the best method based on Abel-Poisson integral and Iran topography conditions, first, we solve the problem by doing a simulated problem. To compare different regularization methods, we used relative errors defined as:

\[
\text{Relative Error} = \frac{\|x^{\text{exact}} - x^{\text{reg}}\|}{\|x^{\text{exact}}\|}
\]

Where \( x^{\text{exact}} \) comes directly from simulation and \( x^{\text{reg}} \) comes from solving the problem via aforementioned methods. Based on our results, ART method is the best suited method for downward continuation problem at geoid computations without applying Stokes formula. Finally, ART method was applied for real gravity modulus for geoid computations in geographical region of Iran.

**Key words:** Geoid computation, Downward continuation problem, Abel-Poisson integral, Regularization methods