Formal Local Cohomology Modules and Serre Subcategories

A. Kianezhad; Science and Research Branch, Islamic Azad University
A. J. Taherizadeh*; Kharazmi University
A. Tehranian; Science and Research Branch, Islamic Azad University

Abstract

Let \(( R, \mathfrak{m} )\) be a Noetherian local ring, \( \mathfrak{a} \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \(( R, \mathfrak{m} )\) is a commutative Noetherian local ring, \( \mathfrak{a} \) an ideal of \( R \) and \( M \) is a finitely generated \( R \)-module. For an integer \( i \in \mathbb{N}_0 \), \( H^i_\mathfrak{a}(N) \) denotes the \( i \)-th local cohomology module of \( M \) with respect to \( \mathfrak{a} \) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \( \{ H^i_\mathfrak{m}(M/\mathfrak{a}^nM) \}_{n \in \mathbb{N}} \) for a non-negative integer \( i \in \mathbb{N}_0 \). With natural homomorphisms, this family forms an inverse system. Schenzel introduced the \( i \)-th formal local cohomology of \( M \) with respect to \( \mathfrak{a} \) in the form of \( f^i_\mathfrak{a}(M) = \lim_{\mathfrak{n} \to \mathfrak{m}} H^i_\mathfrak{m}(M/\mathfrak{a}^nM) \), which is the \( i \)-th cohomology module of the \( \mathfrak{a} \)-adic completion of the Čech complex \( \check{\mathcal{C}}_R^i(M) \), where \( x \) denotes a system of elements of \( R \) such that \( Rad(x) \cap R = \mathfrak{m} \) (see [3, Definition 3.1]). He defines the formal grade as \( f.grade(\mathfrak{a}, M) = \inf \{ i \in \mathbb{N}_0 \mid f^i_\mathfrak{a}(M) \neq 0 \} \). For any ideal \( \mathfrak{a} \) of \( R \) and finitely generated \( R \)-module \( M \) the following statements hold:

(i) (See [3, Theorem 3.11]). If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of finitely generated \( R \)-modules, then there is the following long exact sequence:

\[
\cdots \to f^i_\mathfrak{a}(M') \to f^i_\mathfrak{a}(M) \to f^i_\mathfrak{a}(M'') \to \cdots
\]

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*Corresponding author: taheri@khu.ac.ir
(ii) (See [3, Theorem 1.3]). \( f. \text{grade}(a, M) \leq \dim(M) - cd(a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then \( \text{Hom}_R(\frac{R}{m}, f_a^t(\Gamma_a(M))) \) and \( \text{Hom}_R(\frac{R}{m}, f_a^{t-1}(\frac{M}{\Gamma_a(M)})) \) belong to \( S \), where \( t = f. \text{grade}_S(a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{cd}_S(a, M) \). (See definition 3.1). The main result of this section is that if \( f_a^i(M) \in S \) and \( H^i_M(M) \in S \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^t(M) \) belongs to \( S \).

2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( a \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \).

**Proposition 2.2.** Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

(a) \( f. \text{grade}_S(a, M) \geq \min\{f. \text{grade}_S(a, L), f. \text{grade}_S(a, N)\} \).

(b) \( f. \text{grade}_S(a, L) \geq \min\{f. \text{grade}_S(a, M), f. \text{grade}_S(a, N) + 1\} \).

(c) \( f. \text{grade}_S(a, N) \geq \min\{f. \text{grade}_S(a, L) - 1, f. \text{grade}_S(a, M)\} \).

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[ \cdots \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to f_a^{i+1}(L) \to \cdots \]

So, the result follows.
Corollary 2.3. If $x = x_1, ..., x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( \frac{a^M}{\Sigma^M M} \right) \geq f.\text{grade}_S (a, M) - n$.

Proof. Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \to \frac{M}{(x_1, ..., x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, ..., x_{n-1})M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, ..., x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, ..., x_{n-1})M$ we means 0.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S (a \cap b, M) \geq \min\{f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1\}$.

(b) $f.\text{grade}_S ((a, b), M) \geq \min\{f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M)\}$.

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{a^n M \cap b^n M} \to \frac{M}{a^n M} \oplus \frac{M}{b^n M} \to \frac{M}{(a^n, b^n)M} \to 0.$$

By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \lim_{n \in \mathbb{N}} H^1_m \left( \frac{M}{(a^n b^n)M} \right) \to \lim_{n \in \mathbb{N}} H^1_m \left( \frac{M}{a^n M} \right) \oplus \lim_{n \in \mathbb{N}} H^1_m \left( \frac{M}{b^n M} \right) \to \lim_{n \in \mathbb{N}} H^1_m \left( \frac{M}{(a, b)^n M} \right) \to \cdots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min\{f.\text{grade}_S \left( a, \frac{M}{N_1} \right), f.\text{grade}_S \left( a, \frac{M}{N_2} \right), f.\text{grade}_S (a, M) - f.\text{grade}_S (a, M) + N_1 + N_2 + 1\}$.

(b) $f.\text{grade}_S \left( a, \frac{M}{N_1 + N_2} \right) \geq \min\{f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (a, M)\}.$

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, m)$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, $\inf \left\{ i \mid H^i_m (L) \not\in S \right\} \leq \inf \left\{ i \mid H^i_m (M) \not\in S \right\}.$

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Proof. Let \( L \) be a pure submodule of \( M \). So \( \frac{L}{a^nL} \to \frac{M}{a^nM} \) is pure for each \( n \in \mathbb{N} \). Now according to [8, Corollary 3.2 (a)], \( H^i_m\left( \frac{L}{a^nL} \right) \to H^i_m\left( \frac{M}{a^nM} \right) \) is injective. Since inverse limit is a left exact functor, \( f^i_a(L) \) is isomorphic to a submodule of \( f^i_a(M) \). Consequently, \( f.\text{grade}_{S}(a, L) \geq f.\text{grade}_{S}(a, M) \). If \( a = 0 \) then, \( f.\text{grade}_{S}(0, M) = \inf \{ i | H^i_m(M) \notin S \} \) and the result follows.

Corollary 2.7. If \( 0 \to L \to M \to N \to 0 \) is a pure exact sequence of finitely generated \( R \)-modules, then \( \min \{ f.\text{grade}_{S}(a, L), f.\text{grade}_{S}(a, N) + 1 \} \geq f.\text{grade}_{S}(a, M) \).

Proof. Since \( L \) is a pure submodules of \( M \), as a result of the previous theorem, \( f.\text{grade}_{S}(a, L) \geq f.\text{grade}_{S}(a, M) \). Hence we must prove that \( f.\text{grade}_{S}(a, N) + 1 \geq f.\text{grade}_{S}(a, M) \). We assume that \( i < f.\text{grade}_{S}(a, M) \) and we show that \( i < f.\text{grade}_{S}(a, N) + 1 \). Consider the following long exact sequence:

\[
\cdots \to f^{i-1}_a(M) \to f^{i-1}_a(N) \to f^{i-1}_a(L) \to f^i_a(M) \to f^i_a(N) \to \cdots (**)
\]

If \( i < f.\text{grade}_{S}(a, M) \), then \( f^0_a(M), f^1_a(M), \ldots, f^{i-1}_a(M), f^i_a(N) \in S \). On the other hand, since \( i < f.\text{grade}_{S}(a, L) \), \( f^0_a(L), f^1_a(L), \ldots, f^i_a(L) \in S \). Hence, it follows from (***) that \( f^0_a(N), \ldots, f^{i-1}_a(N) \in S \) and so \( i - 1 < f.\text{grade}_{S}(a, N) \).

Theorem 2.8. Let \(( R, m )\) be a local ring, \( a \) be an ideal of \( R \), \( S \) be a Serre subcategory of the category of \( R \)-modules and \( \text{R homomorphisms and } M \in S \) be a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( \text{Hom}_R\left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S \), where \( t = f.\text{grade}_{S}(a, M) \).

Proof. Due to the previous theorem, \( f.\text{grade}_{S}(a, \Gamma_a(M)) \geq f.\text{grade}_{S}(a, M) \). If \( f.\text{grade}_{S}(a, \Gamma_a(M)) > f.\text{grade}_{S}(a, M) \), then the result is obvious. Accordingly, we assume that \( f.\text{grade}_{S}(a, \Gamma_a(M)) = f.\text{grade}_{S}(a, M) \). We know that \( \text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a) \). By using [4, Lemma 2.3], \( f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M)) \) for all \( i \geq 0 \). So, if \( j < f.\text{grade}_{S}(a, M) \), then \( f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M)) \in S \) and \( \text{Ext}^k_R\left( \frac{R}{m}, f^i_a(\Gamma_a(M)) \right) \in S \) for all \( k \geq 0 \) and \( j < f.\text{grade}_{S}(a, M) \). Moreover \( \text{Ext}^k_R\left( \frac{R}{m}, \Gamma_a(M) \right) \in S \), because \( \Gamma_a(M) \in S \). Consequently, according to [7, Theorem 2.2],

\[
\text{Hom}_R\left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S, \text{ where } t = f.\text{grade}_{S}(a, M).
\]

Corollary 2.9 With the same notations as Theorem 2.8, let \( X \in S \) be a submodule of \( f^i_a(\Gamma_a(M)) \), where \( t = f.\text{grade}_{S}(a, M) \). Then \( \text{Hom}_R\left( \frac{R}{m}, \frac{f^i_a(\Gamma_a(M))}{X} \right) \in S \).

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R\left(\frac{R}{m}, f_a^t(\Gamma_a(M))\right) \to \text{Hom}_R\left(\frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{x}\right) \to \text{Ext}_R^1\left(\frac{R}{m}, X\right)$. (*)

Theorem 2.10. Suppose that $a$ is an ideal of $(R, m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$, where $t = f. \text{grade}_S(a, M)$.

Proof. One has $f. \text{grade}_S(a, \Gamma_a(M)) \geq f. \text{grade}_S(a, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_a(M) \to M \to \frac{M}{\Gamma_a(M)} \to 0$ induces the following long exact sequence:

$$
\cdots \to f_a^{t-1}(\Gamma_a(M)) \xrightarrow{\alpha} f_a^{t-1}(M) \xrightarrow{\beta} f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \xrightarrow{\gamma} f_a^{t-1}(\Gamma_a(M)) \xrightarrow{\delta} \cdots.
$$

Using the exact sequence (*), we obtain the short exact sequence $0 \to \text{Im}(\beta) \to f_a^{t-1}(M) \to \text{Im}(\gamma) \to 0$. Since $f_a^{t-1}(M) \in S$, $\text{Im}(\beta) \in S$ and $\text{Im}(\gamma) \in S$. Furthermore, we have the exact sequence $0 \to \text{Im}(\xi) \to H_m^t(\Gamma_a(M)) \to \text{Im}(\varphi) \to 0$ which induces the following long exact sequence:

$$
0 \to \text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \to \text{Hom}_R\left(\frac{R}{m}, H_m^t(\Gamma_a(M))\right) \to \cdots.
$$

Thus $\text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \in S$. Finally, by considering the short exact sequence $0 \to \text{Im}(\gamma) \to f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \to \text{Im}(\xi) \to 0$ we can conclude that $\text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$.

Theorem 2.11. Suppose that $R$ is complete with respect to the $a$-adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f_a^i(M) \in S$ for all $i < t$. Then $\text{Hom}_R\left(\frac{R}{m}, f_a^t(M)\right) \in S$.

Proof. We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$
\text{Hom}_R\left(\frac{R}{m}, f_a^0(M)\right) \cong \lim_{\rightarrow} \text{Hom}_R\left(\frac{R}{m}, H_m^0(\frac{M}{a^iM})\right) \cong \lim_{\rightarrow} \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^iM}\right)
$$

$$
\cong \text{Hom}_R\left(\frac{R}{m}, \lim_{\rightarrow} \frac{M}{a^iM}\right) \cong \text{Hom}_R\left(\frac{R}{m}, M^0\right) \cong \text{Hom}_R\left(\frac{R}{m}, M\right)
$$

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It is clear that $Hom_R \left( \frac{R}{m}, M \right) \in S$. So by the above isomorphisms, we deduce that

$Hom_R \left( \frac{R}{m}, f_a^0(M) \right) \in S$.

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f_m^t(M)$. Then $f_a^i(M) \cong f_a^i \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M - regular$ element $x \in m$. The exact sequence $0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

\[
\cdots \to f_a^{t-2}(M) \xrightarrow{x} f_a^{t-2}(M) \xrightarrow{f} f_a^{t-2} \left( \frac{M}{xM} \right) \to f_a^{t-1}(M) \xrightarrow{g} f_a^{t-1} \left( \frac{M}{xM} \right) \to f_a^t(M) \xrightarrow{h} \cdots.
\]

Using the exact sequence (*) we obtain the short exact sequence

\[
0 \to f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1} \left( \frac{M}{xM} \right) \to (0 : x) \to 0.
\]

Now, this exact sequence induces the following long exact sequence:

\[
0 \to Hom_R \left( \frac{R}{m}, f_a^{t-1}(M) \right) \to Hom_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \to Hom_R \left( \frac{R}{m}, (0 : x) \right) \to Ext_R^1 \left( \frac{R}{m}, f_a^{t-1}(M) \right) \to \cdots.
\]

By using (*), $f_a^i \left( \frac{M}{x} \right)$ \in $S$ for all $i < t - 1$. Therefore by the induction hypothesis $Hom_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \in S$. Furthermore $Ext_R^1 \left( \frac{R}{m}, f_a^{t-1}(M) \right) \in S$ because $f_a^{t-1}(M) \in S$. Thus in accordance with (**), $Hom_R \left( \frac{R}{m}, (0 : x) \right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

\[
Hom_R \left( \frac{R}{m}, (0 : x) \right) \cong Hom_R \left( \frac{R}{m}, Hom_R \left( \frac{R}{xR}, f_a^t(M) \right) \right) \cong Hom_R \left( \frac{R}{m} \otimes_R \frac{R}{xR}, f_a^t(M) \right) \cong Hom_R \left( \frac{R}{m}, f_a^t(M) \right).
\]

Consequently $Hom_R \left( \frac{R}{m}, f_a^t(M) \right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$, 
$$\sup\{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\} = \dim \left( \frac{M}{a^i} \right).$$

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f \cdot \text{cd}_S(a,M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f \cdot \text{cd}_S(a,L) \leq f \cdot \text{cd}_S(a,N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f \cdot \text{cd}_S(a,N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{a^i} \right) + f \cdot \text{cd}_S(a,N)$, $f^i_a(L) = 0 \in S$. Let $i > f \cdot \text{cd}_S(a,N)$ and the result is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \rightarrow L_{i-1} \rightarrow L_i \rightarrow \frac{L_i}{L_{i-1}} \rightarrow 0$ ($i = 0, l, \ldots, l$). We may assume that $l = l$. The exact sequence $0 \rightarrow K \rightarrow \bigoplus_{j=1}^{l} N \rightarrow L \rightarrow 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

$$\cdots \rightarrow f^i_a(\bigoplus_{j=1}^{l} N) \rightarrow f^i_a(L) \rightarrow f^{i+1}_a(K) \rightarrow \cdots. \ (*)$$

Based on the induction hypothesis, $f^i_a(K) \in S$. Moreover $f^i_a(\bigoplus_{j=1}^{l} N) = \bigoplus_{j=1}^{l} f^i_a(N) \in S$ for all $i > f \cdot \text{cd}_S(a,N)$. Hence it follows from the exact sequence $(*)$ that $f^i_a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f \cdot \text{grade}_S(a,M) = f \cdot \text{grade}_S(a,N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, \mathfrak{m})$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = l$. Then by using [5,Theorem 1.1], $f \cdot \text{grade}_S(a,R) = 1$ and $f \cdot \text{grade}_S(a, \frac{R}{\mathfrak{m}}) = 0$. Set $M := R \bigoplus \frac{R}{\mathfrak{m}}$.

Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But $f \cdot \text{grade}_S(a,M) = \inf\{f \cdot \text{grade}_S(a,R), f \cdot \text{grade}_S(a, \frac{R}{\mathfrak{m}})\} = 0$.

**Corollary 3.4.** For all $x \in a \cdot f \cdot \text{cd}_S(a,M) \geq f \cdot \text{cd}_S(a, \frac{M}{xM})$.

**Corollary 3.5.** Suppose that $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely generated $R$-modules. Then $f \cdot \text{cd}_S(a,M) = \max\{f \cdot \text{cd}_S(a,L), f \cdot \text{cd}_S(a,N)\}$. 
**Proof.** Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, L)$ and $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, N)$. Therefore $f \cdot \text{cd}_S(a, M) \geq \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$.

Next we prove that $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

Let $i > \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$. Then $f_a^i(N), f_a^i(L) \in S$ and from the exact sequence $f_a^i(L) \to f_a^i(M) \to f_a^i(N)$ we conclude that $f_a^i(M) \in S$. Thus, $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $\mathcal{S}$ is defined as
\[
\text{cd}_S(a, M) := \sup \{i \in \mathbb{N} \mid \text{H}_a^i(M) \notin S\}.
\]

The following lemma shows that when we considering the Artinianness of $f_a^i(M)$, we can assume that $M$ is $a$-torsion-free.

**Lemma 3.6.** Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H_m^i(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f_a^i(M) \in S$ for all $i \geq t$.

(b) $f_a^i \left( \frac{M}{\Gamma_a(M)} \right) \in S$ for all $i \geq t$.

**Proof.** According to the hypothesis $t > \text{cd}_S(m, M)$. On the other hand $\text{Supp}_R(\Gamma_a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(m, \Gamma_a(M)) \leq \text{cd}_S(m, M)$. Thus, $t > \text{cd}_S(m, \Gamma_a(M))$ and $H_m^i(\Gamma_a(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:
\[
\cdots \to f_a^i(\Gamma_a(M)) \to f_a^i(M) \to f_a^i \left( \frac{M}{\Gamma_a(M)} \right) \to f_a^{i+1}(\Gamma_a(M)) \to \cdots \ .
\tag{*}
\]

According to [4, Lemma 2.3] $f_a^i(\Gamma_a(M)) \cong H_m^i(\Gamma_a(M))$. By using the hypothesis $f_a^i(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(*)$ that $f_a^i(M) \in S$ if and only if $f_a^i \left( \frac{M}{\Gamma_a(M)} \right) \in S$ for all $i \geq t$.

**Theorem 3.7.** Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(m, M) \leq f \cdot \text{cd}_S(a, M)$. Then $f_a^i(M) \in S$ where $t = f \cdot \text{cd}_S(a, M)$.

**Proof.** We use induction on $d = \dim (M)$. If $d = 0$, then $\dim \left( \frac{M}{aM} \right) = 0$. Accordingly to [3, Theorem 1.1], $f_a^i(M) = 0$ for all $i > 0$.
Moreover $f_a^0(M) \cong M \in S$. By definition $H_i^i(M) \in S$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is a-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \to f_a^i(M) \xrightarrow{x} f_a^i \left( \frac{M}{xM} \right) \xrightarrow{f} f_a^{i+1}(M) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots. (*)$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f.cd_S(a, M)$). So using the above long exact sequence $f_a^i \left( \frac{M}{xM} \right) \in S$ for all $i > t$. By induction hypothesis, $\frac{f_a^i(M)}{a f_a^i(M)} \in S$ because $\dim \left( \frac{M}{xM} \right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence.

$$0 \to \text{Im}(f) \to f_a^i \left( \frac{M}{xM} \right) \to \text{Im}(g) \to 0$$

So we obtain the following long exact sequence.

$$\ldots \to \text{Tor}_R^i \left( \frac{R}{a}, \text{Im}(g) \right) \to \text{Im}(f) \to f_a^i \left( \frac{M}{xM} \right) \to \text{Im}(g) \to 0.$$

Since $f_a^i(M) \in S$ and $\text{Im}(g)$ is a submodule of $f_a^{i+1}(M)$, we deduce that $\text{Tor}_R^i \left( \frac{R}{a}, \text{Im}(g) \right) \in S$. On the other hand, $\frac{f_a^i(M)}{a f_a^i(M)} \in S$. Therefore, $\frac{\text{Im}(f)}{a \text{Im}(f)} \in S$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^i(M)}{a f_a^i(M)} \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^{i+1}(M) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots.$$

So, $\frac{f_a^i(M)}{a f_a^i(M)} \cong \frac{\text{Im}(f)}{a \text{Im}(f)}$ because $x \in a$. Consequently, $\frac{f_a^i(M)}{a f_a^i(M)} \in S$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f.cd_S(a, M) = \max \{ f.cd_S(a, R_P) | P \in \text{Ass}_R(M) \}.$$

**Proof.** Set $N := \mathop{\oplus}_{P \in \text{Ass}_R(M)} R_P$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cd_S(a, M) = f.cd_S(a, N) = \max \{ f.cd_S(a, R_P) | P \in \text{Ass}_R(M) \}$.

**Proposition 3.9.** Assume that $a$ is an ideal of the local ring $(R, m)$. Then $\text{Hom}_R(R_R, f_a^0(M)) \in S$ if and only if $\text{Hom}_R(R_R, \widehat{M}_a) \in S$.

**Proof.** It is enough to consider the following isomorphisms

$$\text{Hom}_R \left( \frac{R_R}{m}, f_a^0(M) \right) \cong \lim_{\substack{\rightarrow \mathbb{N}}} \text{Hom}_R \left( \frac{R_R}{m}, f_a^0 \left( \frac{M}{a^n M} \right) \right) \cong \lim_{\substack{\rightarrow \mathbb{N}}} \text{Hom}_R \left( \frac{R_R}{m}, \widehat{M}^a \right).$$

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