Some properties of n-capable and n-perfect groups

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Abstract

In this article we introduce the notion of n-capable groups. It is shown that every group G admits a uniquely determined subgroup (\([Z^n]^* G\)) which is a characteristic subgroup and lies in the n-centre subgroup of the group G. This is the smallest subgroup of G whose factor group is n-capable. Moreover, some properties of n-central extension will be studied.

Keywords: n-central; n-capable; n-perfect; n-unicentral.

Introduction

In 1979 Fay and Waals [3] introduced the notion of the n-potent and the n-centre subgroups of a group G, for a positive integer n, respectively as follows:

\[ G_n = \langle [x, y^n] | x, y \in G \rangle, \]

\[ Z^n(G) = \{ x \in G | xy^n = y^n x, \forall y \in G \}, \]

where \([x, y^n] = x^{-1}y^{-n}xy^n\). It is easy to see that \(G_n\) is a fully invariant subgroup and \(Z^n(G)\) is a characteristic subgroup of group G. In the case \(n = 1\), these subgroups will be \(G'\) and \(Z(G)\), the drive and centre subgroups of G, respectively. If \(G_n = G\), then G is said to be n-perfect. Let H be a subgroup of G, then \([H, G^n]\) is defined as follows:

\[ [H, G^n] = \langle [h, g^n] | h \in H, g \in G \rangle, \]

and in particular if \(H = G\), we get \(G_n\) The following lemma is similar to the Lemma 2.1 of [5].

Lemma 0.1. Let G and H be two groups and N be a normal subgroup of G. Then

\[ (i) G_n = \{ 1 \} \iff Z^n(G) = G, \]

\[ (ii) (G / N)_n = G_n N / N, \]

\[ (iii) N \subseteq Z^n(G) \iff [N, G^n] = 1, \]

\[ (iv) Z^n(G \times H) = Z^n(G) \times Z^n(H). \]

Materials and Methods

1. n-capability

Baer [1] initiated an investigation of the question "which conditions a group G must be fulfill in order to be isomorphic with the group of inner automorphisms of a group E? As \(\text{InnE} \cong E/Z^n(E)\), it is equivalent to study when \(G \cong E/Z(E)\). By Hall and Senior [4] such a group is called capable. Let n be a positive integer, this notion can be generalized as follows:

Definition 1.1. A group G is said to be n-capable if there exists a group E such that \(G \cong E/Z(E)\). Consider the homomorphism \(\psi : E \rightarrow G\) such that \(Z^n(E)\) includes the kernel of \(\psi\). The intersection of all subgroups of G of the form \(\psi(Z^n(E))\), for every such \(\psi\), denoted by \((Z^n)^*(G)\).

The group G is said to be n-unicentral if \((Z^n)^*(G) = Z^n(G)\). It is easy to see that \((Z^n)^*(G)\) is a characteristic

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subgroup of \( G \) included in \( Z^n(G) \), see [6].

The following theorem is useful in the sense that the quotient group \( G \) by \((Z^n)^*(G)\) is \( n \)-capable which is a generalized version of the work of Beyl, Feglner and Schmid in [2] and similar to the work of Mirebrahimi and Mashayekhy [7] in the case of varieties of groups, see also [8] for more investigations.

**Theorem 1.2.** Let \( H_i \) be normal subgroup of \( G \) and \( G/H_i \) be \( n \)-capable \((i \in I)\). If \( N = \cap_{i \in I} H_i \), then \( G/N \) is \( n \)-capable.

**Proof.** By definition of \( n \)-capability, for any \( i \in I \), there exists the following short exact sequence

\[
1 \to Z^n(E_{i}) \overset{\psi_i}{\to} E_{i} \to G/H_i \to 1.
\]

Let \( B = \prod_{i \in I} Z^n(E_{i}) \), and

\[
A = \{(e_{i}) \in \prod_{i \in I} E_{i}, \quad \exists g \in G \text{ s.t. } \Psi_{i}(e_{i}) = gH_{i}, \forall i \in I \}.
\]

Where \( \prod_{i \in I} X_i \) is the cartesian product of the groups \( X_i \)'s. Clearly \( B \subseteq A \). For any \( g \in G \), we can choose the elements \( e_{gi} \) such that \( \Psi_{i}(e_{gi}) = gH_{i} \). Thus \( e_{g} = (e_{gi}) \in \prod_{i \in I} E_{i} \). Also it is clear that the map

\[
G / N \to A / B,
\]

\[
gN \to e_{g} B
\]

is an isomorphism. Now, as \( B = Z^n(A) \), we conclude that \( G/N \) is \( n \)-capable.

**Theorem 1.3.** \((Z^n)^*(E)\) is the least subgroup lies in the \( n \)-centre of \( G \) such that \( G/(Z^n)^*(G) \) is an \( n \)-capable group.

**Proof.** Let \( 1 \to K \to E \xrightarrow{\psi} G \to 1 \) be an \( n \)-central extension by \( G \), i.e. \( K \subseteq Z^n(E) \).

By isomorphism and Theorem 2.2\( z \) it is clear that \( G/(Z^n)^*(G) \) is \( n \)-capable. Now let \( N \) be a normal subgroup of \( G \), where \( G/N \) is \( n \)-capable. Therefore, there exists an \( n \)-central extension

\[
1 \to Z^n(H) \to H \overset{\phi}{\to} G / N \to 1.
\]

Let \( E = \{(g,h) \in G \times H | gN = \phi (h)\} \) and \( \phi \) be the projection map \( (g,h) \to g \). Then

\[
1 \to Ker \phi \to E \xrightarrow{\phi} G \to 1
\]

is \( n \)-central extension, since \( Z^n(G \times H) = Z^n(G) \times Z^n(H) \). Let \( (g,h) \in Z^n(E) \), \((g_{i},h_{i}) \in G \times H \) such that \( \phi(h_{i}) = g_iN \). Thus, we have

\[
(1,1) = [(g,h), (g_{i},h_{i})]^n = ([g,g_{i}^{n}], [h,h_{i}^{n}]).
\]

Therefore \([h,h_{i}^{n}] = 1, \forall h_{i} \in H \) and then \( h \in Z^n(H) \).

Now we have \( \phi(Z^n(E)) \subseteq N \). Thus by the definition \((Z^n)^*(G) \subseteq \phi(Z^n(E)) \subseteq N \), which completes the proof.

An immediate necessary and sufficient condition for a group \( G \) to be \( n \)-capable is,

**Corollary 1.4.** A group \( G \) is \( n \)-capable if and only if \((Z^n)^*(G) = 1 \).

Now we have a sufficient condition for \( n \)-capability of a group.

**Corollary 1.5.** Let \( N \) be a normal subgroup of \( G \), such that \( N/(Z^n)^*(G) = 1 \). If \( G/N \) is \( n \)-capable, then so is \( G \).

The next theorem shows that the class of \( n \)-capable groups is closed under the direct product which generalizes Proposition 6.3 of [2]. A group \( G \) is said to be subdirect product of the groups \( \{G_i\}_{i \in I} \) if \( G \) is a subgroup of the (unrestricted) direct product \( \prod_{i \in I} G_i \) such that \( p_i(G) = G_i, i \in I \), where \( p_i \)'s are natural projections.

**Theorem 1.6.** Let \( G \) be a subdirect product of the \( n \)-capable groups \( \{G_i\}_{i \in I} \). Then so is \( G \).

**Proof.** Since \( G_i \) is \( n \)-capable, we have the following short exact sequences,

\[
1 \to Z^n(E_{i}) \overset{\psi_i}{\to} E_{i} \xrightarrow{\psi_i} G_{i} \to 1, \quad i \in I.
\]

Define

\[
\Psi = \{\psi_{i} \}_{i \in I} : \prod_{i \in I} E_{i} \xrightarrow{\psi_{i}} G_{i} \to 1.
\]

and let \( E = \psi^{-1}(G), A = \prod_{i \in I} Z^n(E_{i}) \). Then \( A \) is the \( n \)-central subgroup of \( \prod_{i \in I} E_{i} \). Hence we obtain the following commutative diagram,

\[
\begin{array}{cccccc}
1 & \to & A & \to & E & \xrightarrow{\psi} & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \prod_{i \in I} Z^n(E_{i}) & \to & \prod_{i \in I} E_{i} & \xrightarrow{\psi} & \prod_{i \in I} G_{i} & \to & 1,
\end{array}
\]

where \( \psi \) is the restricted map of \( \Psi \) and the vertical maps \( E \to \prod_{i \in I} E_{i} \) and \( G \to \prod_{i \in I} G_{i} \) are inclusions. Since \( G \) is a subdirect product and \( \ker \psi \subseteq E \), the group \( E \) is a subdirect product of \( \{E_i\}_{i \in I} \).

Now it is obvious that \( A \subseteq Z^n(E) \). For the reverse inclusion, let \( (e_i)_{i \in I} \in Z^n(E) \) and \( t_j \in E_i \) for an arbitrary fixed group \( E_i \). Denote also \( p_i \) to be the natural projection for \( E \). Therefore, there exists \( \{t_{ij} \}_{i \in I} \in E \) such that

\[
(1,1) = [(g,h), (g_{i},h_{i})]^n = ([g,g_{i}^{n}], [h,h_{i}^{n}]).
\]
that $p'_i(t_i)_{i\in I} = t_i$. Thus

$$p'([e_i]_{i\in I},[t_i]_{i\in I}) = p'_i([e_i,t^n_i]_{i\in I}) = p'_i([1]_{i\in I}) = 1. $$

On the other hand,

$$p'([e_i]_{i\in I},[t_i]_{i\in I}) = [p'_i([e_i]_{i\in I}),p'_i([t_i]^{p_i}_{i\in I})] = [e_i,t^n_i]. $$

Hence, $[e_i,t^n_i] = 1$ and so the reverse inclusion holds. By $A = Z^n(E)$, we get the $n$-capability of $G$, which completes the proof.

The following corollary is immediate.

**Corollary 1.7.** If $\prod_{i\in I} G_i$ is a weak direct product of the groups $[G_i]_{i\in I}$, then $(Z^n)^*([\prod_{i\in I} G_i]) \subseteq \prod_{i\in I}(Z^n)^* G_i$.

### 2. Application of free presentation

The structure of $Z^n(G)$ by any free presentation for the group $G$ is given in [2]. In this section in a similar way, we study the structure of $(Z^n)^* G$. First, we give the following useful lemma.

**Lemma 2.1.** Let $1 \to R \to F \to G \to 1$ be a free presentation of the group $G$, and $1 \to A \to B \to C \to 1$ be an $n$-central extension of a group $C$. If $\alpha : G \to C$ is a homomorphism, then there exists a homomorphism $\beta : F / [R,F^n] \to B$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
1 & \to & R / [R,F^n] \to F / [R,F^n] \to G \to 1 \\
\downarrow \beta & & \downarrow \alpha \\
1 & \to & A \to B \to C \to 1
\end{array}
$$

Where $\pi$ is the natural homomorphism induced by $\pi$ and $\beta$ is the restriction of $\beta$.

**Theorem 2.2.** For any free presentation $1 \to R \to F \to G \to 1$, and every $n$-central extension $1 \to A \to E \to G \to 1$, we have

$$\bar{\theta}(Z^n)^*(F / [R,F^n]) \subseteq \phi(Z^n)^*(E).$$

**Proof.** By Lemma 2.1 and putting $1 \to A \to E \to G \to 1$ instead of the second row in the diagram, there exists a homomorphism $\beta : F / [R,F^n] \to E$ such that the corresponding diagram is commutative. It is easily to check that $E = A \beta(F / [R,F^n])$ and hence, $\beta((Z^n)^*(F / [R,F^n])) \subseteq (Z^n)^*(E)$. Therefore, we get $\phi(\beta(Z^n)^*(F / [R,F^n])) \subseteq \phi((Z^n)^*(E))$, which completes the proof.

The following important result is immediate.

**Corollary 2.3.** For any free presentation $1 \to R \to F \to G \to 1$ of $G$, we have

$$(Z^n)^*(G) = \pi(Z^n)^*(F / [R,F^n]).$$

### 3. $n$–perfect groups

The concept of covering of a central extension by another central extension has been studied in page 92 of [6]. Here we generalize this notion.

**Definition 3.1.** We say that the $n$-central extension $e$ (uniquely) covers $n$-central extension,

$$1 \to A_1 \to H_1 \to G \to 1,$$

If there exists a (unique) homomorphism $\theta : H \to H_1$ such that the following diagram is commutative,

$$
\begin{array}{ccc}
1 & \to & A \to H \to G \to 1 \\
\downarrow \theta & & \downarrow \theta \\
1 & \to & A_1 \to H_1 \to G \to 1
\end{array}
$$

The $n$-central extension $e$ is said to be universal, if uniquely covers any other $n$-central extension by the group $G$.

The following useful lemma can be easily proved.

**Lemma 3.2.** Let $G$ be an $n$-perfect group. Then $1 \to A \to B \to C \to 1$, is a universal $n$-central extension if and only if any $n$-central extension by $G$ splits.

Now, we present the following theorem which states some essential properties of universal $n$-central extension.

**Theorem 3.3.** Let $e_i : 1 \to A_i \to H_{i+1} \to G \to 1$, be $n$-central extensions by the group $G$. Then

(i) If $e_1$ and $e_2$ are universal $n$-central extensions, then there exists a homomorphism $H_1 \to H_2$ such that maps $A_1$ onto $A_2$,

(ii) If $e_i$ is universal $n$-central extension, then $H_i$ and $G$ are $n$-perfect,

(iii) If $1 \to 1 \to H \to G \to 1$, is a universal $n$-central extension, then so is $1 \to 1 \to G \to G \to 1$.

**Proof.**

(i) The proof is easy, see also Lemma 2.10.1(i) of [6].
(ii) Consider the following \( n \)-central extension,

\[
1 \to A_1 \times H_1 / H_{1n} \to H_1 \times H_1 / H_{1n} \to G \to 1,
\]

where \( \psi(a, bH_{1n}) = \phi_1(a), a \in A_1, b \in H_1 \). Now we define the following homomorphisms

\[
\theta_i : H_1 \to H_1 \times H_1 / H_{1n}, i = 1, 2
\]

\[
\theta_1(h) = (h, 1), \quad \theta_2(h) = (h, hH_{1n}), \quad \forall h \in H_1.
\]

Thus \( \psi \circ \theta_i = \phi_1 \), which implies that \( \theta_1 = \theta_2 \). Therefore \( H_1 = H_{1n} \) and so \( G = G_n \).

(iii) By the definition and part (ii), \( G \) and \( H \) are \( n \)-perfect. If \( 1 \to A \to G \xrightarrow{\psi} G \to 1 \) is an \( n \)-central extension of \( A \) by \( G \), then there exists a homomorphism \( \phi : H \to G^* \) such that \( \phi = \psi \circ \alpha \). Also, \( \alpha \circ \phi^{-1} \) is a homomorphism from \( G \) onto \( G^* \) such that \( \psi \circ (\alpha \circ \phi^{-1}) = 1 \). Thus, by Lemma 3.2 the extension splits.

**Results**

In this paper by means of \( n \)-centre of a group we generalize some properties of capability. Furthermore we characterize a least normal subgroup which lies in the \( n \)-centre of a given group. We derive a necessary and sufficient condition for \( n \)-capability of a group, also a sufficient condition for a group to be \( n \)-capable. Moreover we prove that subdirect product of \( n \)-capable groups is \( n \)-capable. Further we present some properties of covering and uniquely covering of an \( n \)-central extension by another \( n \)-central extension.

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**References**

برخی خواص گروه‌های $n$-توان‌های کامل

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چکیده
در این مقاله مفهوم کروه $n$-توان‌های کامل معرفی می‌شود که هر کروه مانند $G$ گروه معین و منحصربه‌فرد به فرم $(Z^n)^*G$ می‌پذیرد. که $Z$ گروه مشخصه است و در زیر گروه $G$ مرکز گروه $G$ قرار می‌گیرد. این کوچکترین زیر گروه $G$ است به قسمی که $G$ خارج قسمتی آن $n$-توان‌ست. علاوه بر این برخی خواص توسیع $n$-مرکزی مورد مطالعه قرار می‌گیرد.

واژه‌های کلیدی: $n$-مرکزی کامل، $n$-توان، $n$-مرکزی یکپارچه.