Some properties of \( n \)-capable and \( n \)-perfect groups

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Abstract

In this article we introduce the notion of \( n \)-capable groups. It is shown that every group \( G \) admits a uniquely determined subgroup \( (\langle Z^n \rangle)^* (G) \) which is a characteristic subgroup and lies in the \( n \)-centre subgroup of the group \( G \). This is the smallest subgroup of \( G \) whose factor group is \( n \)-capable. Moreover, some properties of \( n \)-central extension will be studied.

Keywords: \( n \)-central; \( n \)-capable; \( n \)-perfect; \( n \)-unicentral.

Introduction

In 1979 Fay and Waals [3] introduced the notion of the \( n \)-potent and the \( n \)-centre subgroups of a group \( G \), for a positive integer \( n \), respectively as follows:

\[
G_n = \langle [x, y^n] \mid x, y \in G \rangle, \\
Z^n(G) = \{x \in G \mid xy^n = y^n x, \forall y \in G \},
\]

where \([x, y^n] = x^{-1}y^{-n}xy^n\). It is easy to see that \( G_n \) is a fully invariant subgroup and \( Z^n(G) \) is a characteristic subgroup of group \( G \). In the case \( n = 1 \), these subgroups will be \( G' \) and \( Z(G) \), the drive and centre subgroups of \( G \), respectively. If \( G_n = G \), then \( G \) is said to be \( n \)-perfect.

Let \( H \) be a subgroup of \( G \), then \([H, G^n]\) is defined as follows:

\[
[H, G^n] = \langle [h, g^n] \mid h \in H, g \in G \rangle,
\]

and in particular if \( H = G \), we get \( G_n \). The following lemma is similar to the Lemma 2.1 of [5].

Lemma 0.1. Let \( G \) and \( H \) be two groups and \( N \) be a normal subgroup of \( G \). Then

(1) \( G_1 = \{1\} \iff Z^n(G) = G \),
(2) \((G / N)_n = G_nN / N \),
(3) \( N \subseteq Z^n(G) \iff [N, G^n] = 1 \),
(4) \( Z^n(G \times H) = Z^n(G) \times Z^n(H) \).

Materials and Methods

1. \( n \)-capability

Baer [1] initiated an investigation of the question "which conditions a group \( G \) must be fulfill in order to be isomorphic with the group of inner automorphisms of a group \( E \) ? As \( \text{InnE} \cong E / Z^n(E) \), it is equivalent to study when \( G \cong E / Z^n(E) \). By Hall and Senior [4] such a group is called capable. Let \( n \) be a positive integer, this notion can be generalized as follows:

Definition 1.1. A group \( G \) is said to be \( n \)-capable if there exists a group \( E \) such that \( G \cong E / Z^n(E) \). Consider the homomorphism \( \psi : E \rightarrow G \) such that \( Z^n(E) \) includes the kernel of \( \psi \). The intersection of all subgroups of \( G \) of the form \( \psi(Z^n(E)) \), for every such \( \psi \), denoted by \((Z^n)^*(G)\).

The group \( G \) is said to be \( n \)-unicentral if \((Z^n)^*(G) = Z^n(G) \). It is easy to see that \((Z^n)^*(G) \) is a characteristic
subgroup of $G$ included in $Z^n (G)$, see [6].

The following theorem is useful in the sense that the quotient group $G$ by $(Z^n)^*(G)$ is $n$-capable which is a generalized version of the work of Beyl, Feglinger and Schmid in [2] and similar to the work of Mirebrahimi and Mashayekhy [7] in the case of varieties of groups, see also [8] for more investigations.

**Theorem 1.2.** Let $H_i$ be normal subgroup of $G$ and $G/H_i$ be $n$-capable ($i \in I$). If $N = \cap_{i \in I} H_i$, then $G/N$ is $n$-capable.

**Proof.** By definition of $n$-capability, for any $i \in I$, there exists the following short exact sequence

$$1 \to Z^n(E_i) \xrightarrow{\psi} G \to G/H_i \to 1.$$ 

Let $B = \prod_{i \in I} Z^n(E_i)$, and

$$A = \{ (e_i) \in \prod_{i \in I} E_i \mid \exists g \in G \text{ s.t. } \psi_i (e_i) = gH_i, \forall i \in I \}.$$ 

Where $\prod_{i \in I} X_i$ is the cartesian product of the groups $X_i$'s. Clearly $B \subseteq A$. For any $g \in G$, we can choose the elements $e_{g\psi}$ such that $\psi_i (e_{g\psi}) = gH_i$. Thus $e_{g\psi} = (e_i) \in \prod_{i \in I} E_i$. Also it is clear that the map

$$G / N \to A / B, \quad gN \to e_{g\psi} B$$

is an isomorphism. Now, as $B = Z^n(A)$, we conclude that $G/N$ is $n$-capable.

**Theorem 1.3.** $(Z^n)^*(E)$ is the least subgroup lies in the $n$-centre of $G$ such that $G / (Z^n)^*(G)$ is an $n$-capable group.

**Proof.** Let $1 \to K \to E \xrightarrow{\psi} G \to 1$ be an $n$-central extension by $G$, i.e. $K \subseteq Z^n(E)$.

By isomorphism and Theorem 2.2z it is clear that $G / (Z^n)^*(G)$ is $n$-capable. Now let $N$ be a normal subgroup of $G$, where $G/N$ is $n$-capable. Therefore, there exists an $n$-central extension

$$1 \to Z^n(H) \to H \xrightarrow{\psi} G / N \to 1.$$ 

Let $E = \{ (g, h) \in G \times H \mid gN = \phi (h) \}$ and $\phi$ be the projection map $(g, h) \mapsto g$. Then

$$1 \to Ker \phi \to E \xrightarrow{\phi} G \to 1$$

is $n$-central extension, since $Z^n (G \times H) = Z^n(G) \times Z^n(H)$. Let $(g, h) \in Z^n(E), (g_1, h_1) \in G \times H$ such that $\phi(h_1) = g_1N$. Thus we have

$$\begin{align*}
(1, 1) &= [(g, h), (g_1, h_1)]^n = ([g, g_1^n], [h, h_1^n]).
\end{align*}$$

Therefore $[h, h_n^n] = 1, \forall h_n \in H$ and then $h \in Z^n (H)$.

Now we have $\phi(Z^n(E)) \subseteq N$. Thus by the definition $(Z^n)^*(G) \subseteq \phi(Z^n(E)) \subseteq N$, which completes the proof.

An immediate necessary and sufficient condition for a group $G$ to be $n$-capable is,

**Corollary 1.4.** A group $G$ is $n$-capable if and only if $(Z^n)^*(G) = 1$.

Now we have a sufficient condition for $n$-capability of a group.

**Corollary 1.5.** Let $N$ be a normal subgroup of $G$, such that $N \cap (Z^n)^*(G) = 1$. If $G/N$ is $n$-capable, then so is $G$.

The next theorem shows that the class of $n$-capable groups is closed under the direct product which generalizes Proposition 6.3 of [2]. A group $G$ is said to be subdirect product of the groups $\{ G_i \}_{i \in I}$ if $G$ is a subgroup of the (unrestricted) direct product $\prod_{i \in I} G_i$ such that $p_i (G) = G_i, i \in I$, where $p_i$'s are natural projections.

**Theorem 1.6.** Let $G$ be a subdirect product of the $n$-capable groups $\{ G_i \}_{i \in I}$. Then so is $G$.

**Proof.** Since $G_i$ is $n$-capable, we have the following short exact sequences,

$$1 \to Z^n(E_i) \xrightarrow{\psi_i} E_i \xrightarrow{\psi} G_i \to 1, \quad i \in I.$$ 

Define

$$\psi = \{ \psi_i \}_{i \in I} : \prod_{i \in I} E_i \xrightarrow{\psi_i} \prod_{i \in I} G_i,$$

and let $E = \psi^{-1}(G), A = \prod_{i \in I} Z^n(E_i)$. Then $A$ is the $n$-central subgroup of $\prod_{i \in I} E_i$. Hence we obtain the following commutative diagram,

$$\begin{array}{ccc}
\prod_{i \in I} Z^n(E_i) & \to & \prod_{i \in I} E_i \\
\downarrow & & \downarrow \\
\prod_{i \in I} G_i & \to & 1
\end{array}$$

where $\psi$ is the restricted map of $\psi$ and the vertical maps $E \to \prod_{i \in I} E_i$ and $G \to \prod_{i \in I} G_i$ are inclusions. Since $G$ is a subdirect product and $\ker \psi \subseteq E$, the group $E$ is a subdirect product of $\{ E_i \}_{i \in I}$.

Now it is obvious that $A \subseteq Z^n(E)$. For the reverse inclusion, let $(e_i)_{i \in I} \in Z^n(E)$ and $t_j \in E_i$ for an arbitrary fixed group $E_i$. Denote also $p_i$ to be the natural projection for $E$. Therefore, there exists $(t_i)_{i \in I} \in E$ such
that \( p'(t'_i)_{i \in \mathcal{I}} = t_1 \). Thus

\[
p'(\{e_i\}_{i \in \mathcal{I}}, \{t'_i\}_{i \in \mathcal{I}}) = p'_i(\{e_i, t^n\}_{i \in \mathcal{I}}) = p'(_i(1)_{i \in \mathcal{I}}) = 1.
\]

On the other hand,

\[
p'(\{e_i\}_{i \in \mathcal{I}}, \{t'_i\}_{i \in \mathcal{I}}) = [p'_i(\{e_i\}_{i \in \mathcal{I}}), p'_i(\{t'_i\}_{i \in \mathcal{I}})]
\]

\[
= [e_i, t^n]
\]

Hence, \([e_i, t^n] = 1\) and so the reverse inclusion holds.

By \( A = Z^\bullet(E) \), we get the \( n\)-capability of \( G \), which completes the proof.

The following corollary is immediate.

**Corollary 2.3.** For any free presentation \( 1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1 \) of \( G \), we have

\[
(Z^n)^*(G) = \pi(Z^n)^*(F/[R, F^n]).
\]

### 3. \( n \)– perfect groups

The concept of covering of a central extension by another central extension has been studied in page 92 of [6]. Here we generalize this notion.

**Definition 3.1.** We say that the \( n\)-central extension \( e \) (uniquely) covers \( n\)-central extension,

\[
1 \rightarrow A_1 \rightarrow H_1 \rightarrow G \rightarrow 1.
\]

If there exists a (unique) homomorphism \( \theta : H \rightarrow H_1 \) such that the following diagram is commutative,

\[
\begin{array}{ccc}
1 & \rightarrow & A \\
\downarrow & & \downarrow \alpha \\
1 & \rightarrow & B
\end{array}
\]

\[
\begin{array}{ccc}
 & & \downarrow \beta \\
 & & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \downarrow I_g \\
 & & 1
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & A_1 \\
\downarrow & & \downarrow \theta \\
1 & \rightarrow & H_1
\end{array}
\]

\[
\begin{array}{ccc}
 & & \downarrow I_{H_1} \\
 & & 1
\end{array}
\]

The \( n\)-central extension \( e \) is said to be universal, if uniquely covers any other \( n\)-central extension by the group \( G \).

The following useful lemma can be easily proved.

**Lemma 3.2.** Let \( G \) be an \( n\)-perfect group. Then \( 1 \rightarrow A \rightarrow G \rightarrow 1 \), is a universal \( n\)-central extension if and only if any \( n\)-central extension by \( G \) splits.

Now, we present the following theorem which states some essential properties of universal \( n\)-central extension.

**Theorem 3.3.** Let \( e_i : 1 \rightarrow A_i \rightarrow H_i \rightarrow G \rightarrow 1 \), be \( n\)-central extensions by the group \( G \). Then

(i) If \( e_1 \) and \( e_2 \) are universal \( n\)-central extensions, then there exists a homomorphism \( H_1 \rightarrow H_2 \) such that maps \( A_1 \) onto \( A_2 \).

(ii) If \( e_1 \) is universal \( n\)-central extension, then \( H_1 \) and \( G \) are \( n\)-perfect,

(iii) If \( 1 \rightarrow H \rightarrow G \rightarrow 1 \), is a universal \( n\)-central extension, then so is \( 1 \rightarrow 1 \rightarrow G \rightarrow 1 \).

**Proof.**

(i) The proof is easy, see also Lemma 2.10.1(i) of [6].
(ii) Consider the following $n$-central extension,

$$1 \rightarrow A \times H \rightarrow H_1 \times H_1 \rightarrow G \rightarrow 1,$$

where $\psi(a, bH_1) = \phi_i(a)$, $a \in A_1$, $b \in H_1$. Now we define the following homomorphisms

$$\theta_i: H_1 \rightarrow H_1 \times H_1 / H_1, i = 1, 2$$

$$\theta_i(h) = (h, 1), \quad \theta_2(h) = (h, hH_1), \quad \forall h \in H_1.$$

Thus $\psi \circ \theta_i = \phi_i$, which implies that $\theta_1 = \theta_2$. Therefore $H_1 = H_1$ and so $G = G_n$.

(iii) By the definition and part (ii), $G$ and $H$ are $n$-perfect. If $1 \rightarrow A \rightarrow G \rightarrow G \rightarrow 1$, is an $n$-central extension of $A$ by $G$, then there exists a homomorphism $\varphi: H \rightarrow G'$ such that $\phi = \psi \circ \varphi$. Also, $\alpha \circ \phi^{-1}$ is a homomorphism from $G$ onto $G'$ such that $\psi \circ (\alpha \circ \phi^{-1}) = 1$. Thus, by Lemma 3.2 the extension splits.

Results

In this paper by means of $n$-centre of a group we generalize some properties of capability. Furthermore we characterize a least normal subgroup which lies in the $n$-centre of a given group. We derive a necessary and sufficient condition for $n$-capability of a group, also a sufficient condition for a group to be $n$-capable. Moreover we prove that subdirect product of $n$-capable groups is $n$-capable. Further we present some properties of covering and uniquely covering of an $n$-central extension by another $n$-central extension.

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References

برخی خواص گروه‌های $n$-توانی و $n$-کامل

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چکیده
در این مقاله مفهوم گروه $n$-توانی وال گروه $n$-کامل را معرفی می‌کنیم. نشان می‌دهیم که هر گروه $G$، که در گروه $Z^n$ به فرم $(g_1, g_2, \ldots, g_n)$ نمایش داده می‌شود، هر گروه $G$ مولفه محور $Z^n$ است و در هر گروه $G$ یک گروه $n$-توانی و $n$-کامل می‌باشد. نتایج این مقاله برای برخی خواص توسیعی $n$-مرکزی و اکنون مطالعه قرار می‌گیرد.

واژه‌های کلیدی: $n$-توانی، $n$-کامل، $n$-مرکزی، یک نواخت.