On the Convergence Rate of the Law of Large Numbers for Sums of Dependent Random Variables

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Abstract

In this paper, we generalize some results of Chandra and Goswami [4] for pairwise negatively dependent random variables (henceforth r.v.’s). Furthermore, we give Baum and Katz’s [1] type results on estimate for the rate of convergence in these laws.

Keywords: Negatively dependent random variables; Complete convergence; Strong law of large numbers

1. Introduction and Preliminaries

Let \( \{X_n, n \geq 1\} \) be a sequence of integrable r.v.’s defined on the same probability space.

Chandra and Goswami [4] have proved the following theorem from the arguments of Csorgo et al. [5].

**Theorem CG1.** Let \( \{X_n, n \geq 1\} \) be a sequence of non-negative r.v.’s with finite \( \text{Var}(X_n) \) and \( f(n) \) be an increasing sequence such that \( f(n) > 0 \) for each \( n \) and \( f(n) \to \infty \) as \( n \to \infty \). Put \( S_n = \sum_{i=1}^n X_i \). Assume that

\[
\sup_{n \geq 1} \frac{1}{f(n)} \sum_{i=1}^n X_i = A \text{ (say)} < \infty; \tag{1.1}
\]

and there is a double sequence \( \{\rho_{ij}\} \) of nonnegative reals such that

\[
\text{Var}(S_n) \leq \sum_{i,j=1}^n \rho_{ij} (f(i \lor j))^2 < \infty, \quad (i \lor j) = \max(i, j). \tag{1.2}
\]

and

\[
\sum_{i=1}^n \sum_{j=1}^n \rho_{ij} (f(i \lor j))^2 < \infty, \quad (i \lor j) = \max(i, j). \tag{1.3}
\]

Then \( (f(n))^{-1}[S_n - E(S_n)] \to 0 \) a.s. as \( n \to \infty \).

Nili and Bozorgnia [11] generalized (and corrected) Theorem CG1 for an array of r.v.’s and obtained the following result:

**Theorem NB.** Let \( \{X_{ni}, n \geq 1, i \geq 1\} \) be an array of non-negative r.v.’s with finite \( \text{Var}(X_{ni}) \) and \( \log_a f(n) \), \( \alpha > 1 \) be an increasing sequence. Put \( S_n = \sum_{i=1}^n a_i X_{ni} \), where \( l(x) \) stands for a nondecreasing continuous function with inverse \( l^{-1} \) such that \( l(n) \) is a natural sequence and \( l(n) \to \infty \). Assume that there is a double sequence of nonnegative reals \( \{\rho_{ij}\} \) such that

\[
\text{Var}(S_n) \leq \sum_{i,j=1}^n \rho_{ij} \text{ for each } n \geq 1; \tag{1.4}
\]
and
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} f^{-2}(l^{-1}(i) \lor l^{-1}(j)) < \infty. \]  
(1.5)

Then \( (f(n))^{-1} [S_n - E(S_n)] \to 0 \) completely as \( n \to \infty \), in the sense of Hsu and Robbins [6] (see also page 225 of Stout [12]), and hence, a.s.

The question underlying the present work is how one may refine Theorem CG1 to give more information on the law of \( \{X_n\} \). We recall the classical answer, the strong law of large numbers Baum and Katz [1] for \( p = 2 \) (see [2]). In Section 3 we generalize Theorem CG1 and give Baum-Katz’s [1] type results on estimate for the rate of convergence in these laws.

Chandra and Goswami [4], also proved Theorem CG2, by Theorem CG1, and extended the results of Landers and Rogge [8] for pairwise independent r.v’s.

Theorem CG2. Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise independent random variables such that there is a sequence \( \{B_n\} \) of Borel subsets of \( R^1 \) satisfying the following conditions

(a) \( \sum_{n=1}^{\infty} P(X_n \in B_n) < \infty \);
(b) \( \sum_{n=1}^{\infty} E(X_i I(X_i \in B_n^c)) = o(f(n)) \);
(c) \( \sum_{n=1}^{\infty} (f^{-2}(n) \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}) < \infty \);
(d) \( \sup_{n \geq 1} \sum_{i=1}^{n} E(X_i I(X_i \in B_n^c)) = o(f(n)) \).

Then \( (f(n))^{-1} [S_n - E(S_n)] \to 0 \) almost surely as \( n \to \infty \).

In Section 3 we also extend Theorem CG2 to negative dependence r.v’s.

2. Negative Dependence

Definition 1. ([9]). Random variables \( X_1,...,X_n \) \( n \geq 2 \) are said to be pairwise negatively dependent (henceforth pairwise ND) if
\[ P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j), \]
(2.1)
holds for all \( x_i, x_j \in \mathbb{R} \) \( i \neq j \). It can be shown that (2.1) is equivalent to
\[ P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j), \]
(2.2)
and
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} f^{-2}(l^{-1}(i) \lor l^{-1}(j)) < \infty. \]

(1.5)

Then \( (f(n))^{-1} [S_n - E(S_n)] \to 0 \) completely as \( n \to \infty \), in the sense of Hsu and Robbins [6] (see also page 225 of Stout [12]), and hence, a.s.

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Theorem CG2. Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise independent random variables such that there is a sequence \( \{B_n\} \) of Borel subsets of \( R^1 \) satisfying the following conditions

(a) \( \sum_{n=1}^{\infty} P(X_n \in B_n) < \infty \);
(b) \( \sum_{n=1}^{\infty} E(X_i I(X_i \in B_n^c)) = o(f(n)) \);
(c) \( \sum_{n=1}^{\infty} (f^{-2}(n) \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}) < \infty \);
(d) \( \sup_{n \geq 1} \sum_{i=1}^{n} E(X_i I(X_i \in B_n^c)) = o(f(n)) \).

Then \( (f(n))^{-1} [S_n - E(S_n)] \to 0 \) almost surely as \( n \to \infty \).

In Section 3 we also extend Theorem CG2 to negative dependence r.v’s.

3. Main Results

In the following theorems \( \alpha \geq 1/2 \) and \( r \) is an integer such that \( r = 2\alpha - 2 \) when \( 2\alpha - 2 \) is integer and \( r = [2\alpha - 2] + 1 \) \( \{[x]\} \) is integer part of \( x \) otherwise.

Also in this paper, \( C \) stands for a generic constant, not necessarily the same at each appearance. Put \( S_n = \sum_{i=1}^{n} X_i \).

Theorem 1. Let \( \{X_n, n \geq 1\} \) be a sequence of r.v’s and \( [f(n), n \geq 1] \) be a sequence of positive reals such that for some \( \beta > 1, [\log \rho(f(n))] \) is an increasing sequence. Assume that there is a double sequence \( \{\beta_{ij}\} \) of non-negative reals such that \( \beta_{ij} \) is upper bound for \( \text{Var}(X_i) \) and
\[ \text{Var}(S_n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}. \]
(3.1)

If for some \( \xi < 2\alpha \)
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} (l^{-1}(i) \lor l^{-1}(j))^{-1/2} < \infty, \]
(3.2)
then for every \( \varepsilon > 0 \)
\[ \sum_{n=1}^{\infty} n^{2a-2} P(\|S_n - E(S_n)\| > ef^{a}(n)) < \infty. \] (3.3)

**Remark.** If \( a = 1 \) we can use theorem NB for \( X_n = X_i, I(n) = n \) and \( a_n = 1 \), it is sufficient to replace (3.2) by (1.5), then (3.3) holds.

**Proof.** We use sub-sequence method. Replacing \( X_i \) by \( X - E(X_i) \) we may use \( E(X_i) = 0 \). It is easy to show that

\[
\sum_{n=1}^{\infty} n^{2a-2} P(\|S_n\| > ef^{a}(n^2)) + \sum_{i=1}^{\infty} k^{2a-2} P(\|S_n\| > \varepsilon / f^{a}(k)) + \sum_{i=1}^{\infty} k^{2a-2} P(D_n > \varepsilon / f^{a}(k)) ,
\]

where \( D_n = \max_{n' < k < n} |S_{n' - S_n}| \). It is sufficient to show that each of three above series is convergent.

\[
\sum_{n=1}^{\infty} n^{2a-2} P(\|S_n\| > ef^{a}(n^2)) \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) ,
\]

for a fix \( i \) the second sum include one statement and we have

\[
\sum_{i= \infty}^{\infty} \rho_i \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) \leq C \sum_{i= \infty}^{\infty} \rho_i \sum_{n \in k^{2a-2}} n^{2a-2} \beta(f^{a}(k)) .
\]

Note that \( \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \leq C \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \leq C \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] \left[ \sqrt{i} \right] .

The proofs follow the same lines as the proof of Theorem 1.

**Theorem 2.** Let \( \{X_n, n \geq 1\} \) and \( \{\rho_i\} \) be as in Theorem 1 such that
Let $f(n)$ be an increasing sequence such that \{n/f(n)\} be a bounded sequence. If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} / (i \vee j)^{3/2} < \infty,$$

(3.4)

then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^\alpha(n)) < \infty.$$

**Proof.** The Chebyshev's inequality, condition (3.4) and a change of order of summation imply that

$$\sum_{n=1}^{\infty} n^{2(2\alpha-2)} P(|S_n| > \varepsilon f^\alpha(n^2)) \leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{(n^2)} E(S_n)^2 \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} / n^4 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} \sum_{n=1}^{\infty} 1/n^4 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} / (i \vee j)^{3/2} < \infty.$$

For the second series we have

$$\sum_{n=1}^{\infty} k^{2\alpha-2} P(|S_n| > \varepsilon f^\alpha(k))/2 \leq C \sum_{n=1}^{\infty} \frac{k^{2\alpha-2}}{f^{2\alpha}(k)} E(S_n)^2 \leq C \sum_{n=1}^{\infty} \rho_{kn} \sum_{i=1}^{n} 1/n^4 \leq C \sum_{n=1}^{\infty} \rho_{kn} / (i \vee j)^{3/2} < \infty.$$

And finally we show that the third series is convergent

$$\sum_{n=1}^{\infty} k^{2\alpha-2} P(D_n > \varepsilon / 2f^\alpha(k)) \leq C \sum_{n=1}^{\infty} k^{2\alpha-2} E(D_n^2) \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{2n^{2\alpha-2}}{n^4} \rho_{ij} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} \frac{1}{n^4} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(n+1)^{2\alpha}} \frac{1}{f(i \vee j)^{2\alpha}}$$

$$\leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} \left(\frac{1}{(x+1)^{2\alpha}} f(i \vee j)^{2\alpha}\right) < \infty.$$

**Theorem 3.** Let \{X_n, n \geq 1\} be a sequence of r.v.'s and \{\rho_{ij}\} be a double sequence of nonnegative reals such that

$$Var(S_n) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} \text{ for each } n \geq 1;$$

(3.5)

Assume that \{f(n)\} is an increasing sequence such that $n^\beta \leq f(n) \leq (n+1)^\beta$ for some $0 < \beta \leq 1$ and for each $n \geq 1$. If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} < \infty,$$

where $\gamma = (3+4\alpha\beta - 4\alpha) / 2\beta$ and $\alpha < 3 / 4(1-\beta)$, then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^\alpha(n)) < \infty.$$

**Proof.** Again we are going to use subsequence method. Replacing $X_i$ by $X_i - E(X_i)$, we may assume $E(X_i) = 0$.

$$\sum_{n=1}^{\infty} n^{2(2\alpha-2)} P(|S_n| > \varepsilon f^\alpha(n^2)) \leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)} E(S_n)^2 \leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}$$

$$= C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} \left(\frac{1}{x^{2\beta}} f(i \vee j)^{2\beta}\right) < \infty.$$
\[ \sum_{n=1}^{\infty} \sum_{i=1}^{n} \rho_y(i) \leq C \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \rho_y(i) \left( \frac{1}{(i+j)^{0.5}} \right) < \infty \]

by Chebyshev’s inequality and (3.5). For the second sum we have

\[ \sum_{n=1}^{\infty} k^{2a-2} P\{S_n \geq \varepsilon f^{-a}(k)/2\} \]

Thus it remains to verify that the third sum is convergent

\[ \sum_{n=1}^{\infty} k^{2a-2} P(D_n > \varepsilon / 2f^{-a}(k)) \]

By Theorem 1, conditions (a) and (b) and Proposition 1 applied to \( \{Y_n, n \geq 1\} \)

**Proof.** Put \( Y_n = X_n I(X_n \in B_n) + x_n I(X_n \notin B_n) \), \( Z_n = X_n - Y_n \), \( S_n = \sum_{i=1}^{n} X_i \), \( S'_n = \sum_{i=1}^{n} Y_i \) and \( S''_n = S_n - S'_n = \sum_{i=1}^{n} Z_i \), \( n \geq 1 \). It is obvious that \( \{Y_n, n \geq 1\} \) and \( \{Z_n, n \geq 1\} \) are two sequences of pairwise ND r.v.’s. It is sufficient to show that

\[ \sum_{n=1}^{\infty} n^{2a-2} P\{S''_n - E(S''_n) > \varepsilon f^{-a}(n)\} < \infty, \]

By Theorem 1, conditions (a) and (b) and Proposition 1 applied to \( \{Y_n, n \geq 1\} \) yields

\[ \sum_{n=1}^{\infty} n^{2a-2} P\{S''_n - E(S''_n) > \varepsilon f^{-a}(n)\} \]

\[ \leq C \sum_{n=1}^{\infty} n^{2a-2} \sum_{j=1}^{\infty} E(X_j^2) \]

\[ \leq C \sum_{n=1}^{\infty} n^{2a-2} \sum_{j=1}^{\infty} \frac{1}{(n+1)^{0.5}} \]

**Theorem 4.** Let \( \alpha, \beta, \gamma, \rho, \beta_f \) and \( f(n) \) be as in Theorem 1. Also, let \( \{X_n, n \geq 1\} \) be a sequence of pairwise ND r.v.’s such that there is a sequence \( \{B_n, n \geq 1\} \) of semi intervals \( (\infty, x_n) \) or \( (x_n, \infty) \), \( x_n \in \mathbb{R} \), satisfying in the following conditions:

(a) \( \sum_{n=1}^{\infty} C_n P(X_n \in B'_n) < \infty \]

where

\[ C_n = \frac{\beta_f^{0.5}}{\beta_f^{0.5}} n^{-0.5} ; \]

(b) \( \sum_{n=1}^{\infty} n^{2a-2} E(X_n^2 I(X_n \in B_n)) < \infty \)

(c) \( \{X_n - x_n\} I(X_n \notin B'_n) \) is uniformly integrable;

here \( B'_n \) is the complement of \( B_n \). Then for every \( \varepsilon > 0 \)

\[ \sum_{n=1}^{\infty} n^{2a-2} P\{S_n - E(S_n) > \varepsilon f^{-a}(n)\} < \infty. \]
\[
\sum_{n=1}^{\infty} n^{2a-2} P \left( |S_n' - E(S_n')| > \epsilon f^a(n) \right) \\
\leq C \sum_{n=1}^{\infty} n^{2a-2} E \left( |S_n' - E(S_n')| \right) < \infty,
\]

by (c) and the first Borel Cantelli lemma, the desired result follows.

References