Numerical Solution of Delay Differential Equations by Differential Transform Method

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Abstract

Introduction: The theory and application of linear and nonlinear delay differential equations is an important subject within physics and applied mathematics. There are several numerical approaches for solving linear and nonlinear delay differential equations.

Aim: In this paper, differential transform method (DTM) is applied to numerical solution of linear and nonlinear delay differential equations.

Materials and Methods: In this work, we presented simple proofs of the differential transform Theorems and then we applied them to solving linear and nonlinear delay differential equations.

Results: Numerical results compared to exact solutions are reported and it is shown that DTM is a reliable tool for the solution of delay differential equations.

Conclusion: The present method reduces the computational difficulties of the other traditional methods and all the calculations can be made simple manipulations. Several examples were tested by applying the DTM and the results have shown remarkable performance.

Keywords: Differential transform method; Delay differential equations; Series solution; Numerical solution.

Introduction

Consider the following initial value problem for the nth order delay differential equations **(** DDE **)** of the form

$$\begin{cases} y^{(n)}(x) = f\left(x, y(x), y(\alpha_1(x)), y(\alpha_2(x)), \ldots, y(\alpha_m(x))\right); 0 \le x \le b, \\ \left(y(x), y^{'}(x), \ldots, y^{(n-1)}(x)\right) = \left(g_1(x), g_2(x), \ldots, g_n(x)\right); a \le x \le 0, \end{cases} \tag{1}$$

Where $a = \min\limits_{1 \le i \le m} \alpha_i(x)$; for $x \in [0, b]$, $g(x) = \left(g_1(x), g_2(x), \ldots, g_n(x)\right)$ are the initial functions and for i=1(1)m; $\alpha_i(x) \le x$ are the delay functions. We assume that $\alpha_1, \alpha_2, \ldots, \alpha_m, g_1, g_2, \ldots, g_n$ and f are sufficiently smooth. [8,10]

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In this paper, we extended the DTM for solving equations (l).The basic definition and the fundamental Theorems of the DTM and its applicability for various kinds of differential equations are given in. [1,2,4-7,11,13-15] For convenience of the reader, we will present a review of the DTM.

Materials and Methods

Review of DTM

The differential transform of the **kth** derivative of function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}, \qquad (2)$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function. The differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-x_0)^k. \qquad (3)$$

From equations (2) and (3), we get

$$y(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left.\frac{d^k y(x)}{dx^k}\right|_{x=x_0}, \qquad (4)$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In real applications, the function $y(x)$ is expressed by a finite series and equation (3) can be written as

$$y(x) = \sum_{k=0}^{n} Y(k)(x-x_0)^k, \qquad (5)$$

here **n** is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1.

Table 1- Operations of DTM[1,3]

Original function	Transformed function
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=cu(x), c\in R$	$Y(k)=cU(k)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{l=0}^{k} V(l)U(k-l)$
$y(x)=\dfrac{d^n u(x)}{dx^n}, n\in N$	$Y(k)=\dfrac{(k+n)!}{k!}U(k+n)$
$y(x)=x^n$	$Y(k)=\delta(k-n)$
$y(x)=e^{\lambda x}, \lambda\in R$	$Y(k)=\dfrac{\lambda^k}{k!}$
$y(x)=f(x+a)$	$Y(k)=\sum_{h_1=k}^{N}\binom{h_1}{k}a^{h_1-k}F(h_1)$ for $N\to\infty$
$y(x)=f_1(x)f_2(x)...f_{n-1}(x)f_n(x)$	$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}F_1(k_1)$ $\times F_2(k_2-k_1)...F_{n-1}(k_{n-1}-k_{n-2})F_n(k-k_{n-}$

Main results

The following Theorems provide us the differential transform method of given functions:

Theorem 1. If $y(x)=f(x/a), a\geq 1$, then $Y(k)=\dfrac{1}{a^k}F(k)$.

Proof:

It is easy to obtain the following from $y(x)$

$$\frac{d^n y(x)}{dx^n}=\frac{d^n f(x/a)}{dx^n},$$

then by Table 1 we have

$$\frac{(k+n)!}{k!}Y(k+n)=\frac{1}{a^n}f^{(n)}(x/a),$$

$$\Rightarrow \frac{(k+n)!}{k!}Y(k+n)=\frac{1}{a^n}\frac{(k+n)!}{k!}F(k+n),$$

$$\Rightarrow Y(k+n)=\frac{1}{a^n}F(k+n).$$

Now, set $k\to k-n$, we have $Y(k)=\dfrac{1}{a^n}F(k)$ and set, $n\to k$ we have

$$Y(k)=\frac{1}{a^k}F(k).$$

Theorem 2. If $y(x)=f_1(x/a_1)f_2(x/a_2), a_1\geq 1, a_2\geq 1$, then

$$Y(k)=\sum_{l=0}^{k}\frac{1}{a_2^l}F_2(l)\frac{1}{a_1^{k-l}}F_1(k-l).$$

Proof:

Let the differential transform of $f_1(x/a_1)$ and $f_2(x/a_2)$ be $F_1(k)$ and $F_2(k)$, respectively, by using Table 1 we have $Y(k)=\sum_{l=0}^{k}F_2(l)F_1(k-l).$

then by Theorem 1 we have

$$Y(k)=\sum_{l=0}^{k}\frac{1}{a_2^{l}}F_2(l)\frac{1}{a_1^{k-l}}F_1(k-l).$$

Note 1: If $a_1=a_2=a$ and $F_1(k)=F_2(k)=F(k)$ then,

$$Y(k)=\sum_{l=0}^{k}\frac{1}{a^{k}}F(l)F(k-l).$$

Theorem 3. If $y(x)=f_1(x/a_1)f_2(x/a_2)\dots f_{n-1}(x/a_{n-1})f_n(x/a_n)$, $a_i\geq 1(i=1(1)n)$, then

$$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{1}{a_1^{k_1}}F_1(k_1)\frac{1}{a_2^{k_2-k_1}}F_2(k_2-k_1)\dots\frac{1}{a_{n-1}^{k_{n-1}-k_{n-2}}}$$
$$\times F_{n-1}(k_{n-1}-k_{n-2})\frac{1}{a_n^{k-k_{n-1}}}F_n(k-k_{n-1}).$$

Proof:

Let differential transform of $f_i(x/a_i)$ for $i=1(1)n$, be $F_i(k)$, by using Table 1, we have differential transform of $y(x)$ as

$$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}F_1(k_1)F_2(k_2-k_1)\dots F_{n-1}(k_{n-1}-k_{n-2})$$
$$\times F_n(k-k_{n-1}),$$

then by Theorem 1 we have

$$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{1}{a_1^{k_1}}F_1(k_1)\frac{1}{a_2^{k_2-k_1}}F_2(k_2-k_1)\dots\frac{1}{a_{n-1}^{k_{n-1}-k_{n-2}}}$$
$$\times F_{n-1}(k_{n-1}-k_{n-2})\frac{1}{a_n^{k-k_{n-1}}}F_n(k-k_{n-1}).$$

Note 2: If $a_1=a_2=\dots=a_n=a$ and $F_1(k)=F_2(k)=\dots=F_n(k)=F(k)$

then, $Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{1}{a^{k}}F(k_1)$

$$\times F(k_2-k_1)\dots F(k_{n-1}-k_{n-2})F(k-k_{n-1}).$$

Results and Discussion

In order to illustrate the advantages and the accuracy of the DTM for solving the linear and nonlinear delay differential equations, we have applied the method to different delay differential equations.

Example 1. Consider the following linear delay differential equation:[9,12]

$$\begin{cases} \dfrac{d^2y}{dx^2} = \dfrac{3}{4}y(x) + y(\dfrac{x}{2}) - x^2 + 2; \quad 0 \le x \le 1, \\ \qquad\qquad y(0) = 0, \quad y^{'}(0) = 0, \end{cases} \tag{6}$$

where exact solution of this problem is $y(x) = x^2$.

By Table 1 and Theorem 1, the differential transform for equations (6) as follows:

$$\begin{cases} \dfrac{(k+2)!}{k!}Y(k+2) = \dfrac{3}{4}Y(k) + \dfrac{1}{2^k}Y(k) - \delta(k-2) + 2\delta(k), \\ \qquad Y(0) = 0, \quad Y(1) = 0. \end{cases} \tag{7}$$

Consequently, we find

$Y(2) = 1, \quad Y(k) = 0 \quad \text{for } k \ge 3.$

Then, by using equation (3), we obtain the exact solution $y(x) = x^2$.

Example 2.Consider the following nonlinear delay differential equation [9]:

$$\begin{cases} \dfrac{d^3y}{dx^3} = -1 + 2y^2(\dfrac{x}{2}); \quad 0 \le x \le 1, \\ y(0) = 0, \quad y^{'}(0) = 1, \quad y''(0) = 0, \end{cases} \tag{8}$$

where exact solution of this problem is $y(x) = \sin(x)$.

By Table l and Theorem 2, the differential transform for equations (8) as follows:

$$\begin{cases} \dfrac{(k+3)!}{k!}Y(k+3) = -\delta(k) + 2\sum_{l=0}^{k}\dfrac{1}{2^k}Y(l)Y(k-l), \\ \qquad Y(0) = 0, \quad Y(1) = 1, \quad Y(2) = 0. \end{cases} \tag{9}$$

Consequently, we find

$$Y(k) = \begin{cases} 0, & k = 2n \\ \dfrac{(-1)^n}{(2n+1)!}, & k = 2n+1 \end{cases}, n \in N.$$

Then, by using equation (3), we obtain the exact solution $y(x) = \sin(x)$.

Conclusion

In this work, we successfully apply the DTM to find numerical solutions for linear and nonlinear delay differential equations. It is observed that DTM is an effective and reliable tool for the solution of delay differential equations. The method gives rapidly converging series solutions. The accuracy of the obtained solution can be improved by taking more terms in the solution.

In many cases, the series solutions obtained with DTM can be written in exact closed form. The present method reduces the computational difficulties of the other traditional methods and all the calculations can be made simple manipulations. Several examples were tested by applying the DTM and the results have shown remarkable performance.

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