A revision on area ranking and deviation degree methods of ranking fuzzy numbers

R. Ghasemi*, M. Nikfar and E. Roghania

Department of Industrial Engineering, K.N Toosi University of Technology, Vanak Square, Pardis St, Tehran, P.O. Box 1631711491, Iran.

Received 6 November 2012; received in revised form 23 May 2014; accepted 9 August 2014.

Abstract. Recently two important methods [Wang Zh., X., Liu, Y.J. and Feng, B. “Ranking L-R fuzzy number based on deviation degree”, Information Science, pp. 2070-2077 (2009); and Wang, Y.M. and Liu, Y. “Area ranking of fuzzy numbers based on positive and negative ideal points”, Computers and Mathematics with Applications, pp. 1769-1779 (2009)] have been proposed for ranking fuzzy numbers. But we have found that they both have a same basic disadvantage. In this paper, after a short review on different proposed fuzzy number ranking methods, we explain the drawback on deviation degree and the area ranking methods and provide an improvement method to overcome this shortage. Our approach is based on the maximization set and minimization set methods concepts. The results show the superiority of the proposed method in comparison with other ranking methods, especially when the ranking of the inverse and the symmetry of the fuzzy numbers are of interest.

KEYWORDS
Fuzzy number; Deviation degree; Area ranking; Risk attitude; Maximization set; Minimization set.

© 2015 Sharif University of Technology. All rights reserved.

1. Introduction

Under fuzzy environment, ranking Fuzzy Numbers (FNs) is an important part of decision making process. Following the Zadeh’s paper [1] on fuzzy set theory and then Jain’s paper [2] and Dubois and Prade’s paper [3] on FNs, the fuzzy theory and its application have grown explosively. There exist many different methods in ranking FNs. The earliest method of ranking the FNs was proposed by Jain [4]. For the triangular and trapezoidal FNs, Liu and Wang [5] used the concepts of the integral values for ranking normal and non-normal FNs. Cheng [6] indicated that Liu and Wang’s [5] method has a defect in ranking normal and non-normal triangular/trapezoidal FNs because it consider

the normal and non-normal triangular/trapezoidal FNs equal. Chu and Tsao [7] proposed a new approach for ranking FN that considered the area between the centroid and original points. But, it has been known that their approach has some drawbacks. Also, Wang et al. [8] explained that the centroid formulae provided by Cheng [6] is not always true and leads to some misapplications with some FNs. They gave the revised centroid formulae for ranking FNs. Abbaspour and Asadi [9] performed a modification on distance based methods and proposed a new fuzzy number ranking method, called sign distance method. Asady and Zendehman [10] used the nearest point of support function for ranking FNs. Wang and Lee [11] gave a revision on Chu and Tsao’s approach [7] to overcome its defects. Chu and Lin [12] applied an interval arithmetic method based on fuzzy TOPSIS model for ranking fuzzy numbers. Sedi-Nezhad and Daneshi [13] introduced use of the preference ratio with a moderate modification as an efficient method for ranking negative fuzzy numbers. Ramli and Mohamad [14] used Jaccard

* Corresponding author. Tel: +98 21 84043344; Fax: +98 21 84063340
E-mail addresses: rghasemi@mail.kntu.ac.ir (R. Ghasemi);
M. Nikfar@gmail.com (M. Nikfar);
E_roghanian@kntu.ac.ir (E. Roghania)
similarity measure index with degree of optimism. Xu and Zhai [15] proposed an improved method for ranking FNs by distance minimization. Zhang and Yu [16] proposed a pairwise comparison based method for ranking L-R fuzzy numbers. A review list on different fuzzy number ranking methods was provided in Table 1. For more details, readers are referred to the references.

Wang et al. [17], introduced an approach to ranking L-R fuzzy numbers based on a deviation degree. Also Wang and Luo [18] proposed the positive and the negative ideal point concept for ranking FNs. They defined two new alternative indices for the purpose of ranking. The two new indices are defined for ranking based on idiosyncrasies of a Decision Maker (DM)’s policy towards risks and the left and the right areas between FNs and the two ideal points. Though these two methods acted well, we found that they both have a same basic disadvantage with some FNs. In this paper, we explain this defect and provide an improvement approach to overcome this shortage. This approach is based on considering the maximization and minimization sets concepts and DM’s risk attitudes.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts and definitions of the FNs. Section 3 briefly introduces the two debatable methods and explains their shortage. Section 4 proposes an improved approach to overcome the shortage of these methods. In section 5, some numerical examples are explained to verify the efficiency of the proposed improvement approach. Finally, this study concludes in Section 6.

2. Preliminaries

Definition 1. Let X be a universe set. A fuzzy subset A of X is defined with a membership function \( \mu_A(x) \) mapping each element \( x \) in A to a real number in the interval [0, 1]. The membership function of a FN A is defined as follows (see [8]):

\[
\mu_A(x) = \begin{cases} 
 f_A^L(x) & a \leq x \leq b \\
 1 & b \leq x \leq c \\
 f_A^R(x) & c \leq x \leq d \\
 0 & \text{otherwise}
\end{cases}
\]

(1)

where \( f_A^L(x) : [a, b] \rightarrow [0, 1] \) and \( f_A^R(x) : [c, d] \rightarrow [0, 1] \) are two continuous functions mapping from the real line \( R \) to the closed interval [0, 1]. The former is a strictly increasing function called the left membership function and the latter is a monotonically decreasing function called the right membership function. If \( f_A^L(x) \) and \( f_A^R(x) \) are both linear, then A is referred to as a trapezoidal FN and is usually denoted by \( A = (a, b, c, d) \). In particular, when \( b = c \), the trapezoidal FN is reduced to a triangular FN, denoted by \( A = (a, b, d) \). So, triangular FNs are special cases of trapezoidal FNs. The set of all these fuzzy numbers is denoted by \( E \).

Definition 2. An L-R fuzzy number \( A = (m, n, \alpha, \beta)_{LR} \), \( m \leq n \) and \( \alpha, \beta \geq 0 \) is defined as follows (see [18]):

\[
\mu_A(x) = \begin{cases} 
 L \left( \frac{m-x}{\alpha} \right) & -\infty < x < m \\
 1 & m \leq x \leq n \\
 R \left( \frac{x-n}{\beta} \right) & n < x < \infty
\end{cases}
\]

(2)

Definition 3. For fuzzy set \( A \), the support set of \( A \) is defined as (see [17]):

\[
s(A) = \{ x \in R | \mu_A(x) > 0 \}.
\]

(3)

Definition 4. Let \( x_{min} \) and \( x_{max} \) be the infimum and supremum of the support set of an arbitrary group of L-R fuzzy numbers, \( A_1, A_2, ..., A_n \), respectively. Then \( A_{min} \) and \( A_{max} \) are defined as the minimization set and the maximization set respectively, and their membership function is given by (see [17]):

\[
\mu_{A_{min}} = \begin{cases} 
 \frac{x_{min}-x}{x_{min}-x_{min}} & x \in S \\
 0 & \text{otherwise}
\end{cases}
\]

(4)

and:

\[
\mu_{A_{max}} = \begin{cases} 
 \frac{x-x_{max}}{x_{max}-x_{max}} & x \in S \\
 0 & \text{otherwise}
\end{cases}
\]

(5)

where \( S \) is the support set of these FNs, i.e. \( s = \bigcup_{i=1}^{n} s(A_i) \).

3. The shortage of the area ranking and the deviation degree ranking methods

In this section, we briefly introduce Wang et al.’s [17] method and Wang and Luo’s [18] method. Then we analytically discuss on their shortage.

3.1. Ranking L-R fuzzy numbers based on deviation degree

Wang et al. [17] utilized maximization and minimization sets and the left and right deviation degree concepts in their method.

Definition 5. For a group of L-R fuzzy numbers, \( A_1, A_2, ..., A_n \) the left deviation degree and the right deviation degree of \( A_i (i \in 1, 2, ..., n) \) that is defined by Wang et al. [17] are calculated as follows (see [17] and
<table>
<thead>
<tr>
<th>Paper title</th>
<th>Author(s)</th>
<th>Year</th>
<th>Method for ranking</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>New pairwise comparison based method of ranking L-R fuzzy numbers</td>
<td>Zhang, M. and Yu, F.A. [16]</td>
<td>2011</td>
<td>This paper proposed a new approach for ranking L-R FNs based on pairwise comparison</td>
<td></td>
</tr>
<tr>
<td>On the Jaccard index with degree of optimism in ranking fuzzy numbers</td>
<td>Ramli, N., and Mohamad, D. [14]</td>
<td>2010</td>
<td>The authors used Jaccard similarity measure index with degree of optimism for FN ranking</td>
<td></td>
</tr>
<tr>
<td>Triangular approximations of fuzzy numbers using alpha-weighted valuations</td>
<td>Abbaspour, S., Ahmady, E. and Ahmady, N. [10]</td>
<td>2010</td>
<td>A fuzzy triangular approximation using α-weighted valuations is introduced and the nearest approximation by the minimization technique is obtained</td>
<td></td>
</tr>
<tr>
<td>Ranking L-R fuzzy number based on deviation degree</td>
<td>Wang Zh., X., Liu, Y.J., and Feng, B. [17]</td>
<td>2000</td>
<td>This paper introduced an approach to ranking L-R fuzzy numbers based on deviation degree. In their approach, the maximal and minimal reference sets are constructed to measure the L-R deviation degree of FNs.</td>
<td></td>
</tr>
<tr>
<td>A new method for analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers</td>
<td>Sangamset, K.A.T.A. and Chen, S.M. [20]</td>
<td>2000</td>
<td>The proposed method calculates the areas on the positive side, the areas on the negative side, the spreads and the heights of the generalized fuzzy numbers to evaluate ranking scores of the generalized fuzzy numbers</td>
<td>Generalized FNs</td>
</tr>
<tr>
<td>Application of a fuzzy TOPSIS method base on modified preference ratio and fuzzy distance measurement in assessment of traffic police centers performance</td>
<td>Sadi-Nezhad, S. and Khalili Daneshvar, K. [13]</td>
<td>2000</td>
<td>Preference ratio with a moderate modification for negative fuzzy numbers was used as an efficient ranking method for fuzzy numbers in a relative manner</td>
<td>Generalized fuzzy numbers</td>
</tr>
<tr>
<td>The satisfaction degree of the fuzzy numbers and ranking of the fuzzy numbers</td>
<td>Shi, Y.Y. and Xue-Jian, Y. [21]</td>
<td>2000</td>
<td>In this paper a new concept of the satisfaction degree of the fuzzy number is presented</td>
<td></td>
</tr>
<tr>
<td>Area ranking of fuzzy numbers based on positive and negative ideal points</td>
<td>Wang, Y.M. and Liu, Y. [18]</td>
<td>2000</td>
<td>This paper presents an alternative ranking approach for fuzzy numbers called area ranking based on positive and negative ideal points</td>
<td></td>
</tr>
<tr>
<td>Fuzzy risk analysis based on ranking fuzzy numbers using α-cuts, belief features and signal/noise ratios</td>
<td>Chen, S.M. and Wang, C.H. [22]</td>
<td>2000</td>
<td>This paper proposed a new method for ranking fuzzy numbers using the α-cuts, the belief feature and the signal/noise ratios</td>
<td></td>
</tr>
<tr>
<td>Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads</td>
<td>Chen, S.M. and Chen, J.H. [23]</td>
<td>2000</td>
<td>The proposed method considers the defuzzified values, the heights and the spreads for ranking generalized fuzzy numbers</td>
<td>Generalized fuzzy numbers</td>
</tr>
<tr>
<td>Fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations</td>
<td>Lee, L.W. and Chen, S.M. [25]</td>
<td>2008</td>
<td>This paper represented a new method for ranking trapezoidal fuzzy numbers based on their shapes and deviations</td>
<td>Trapezoidal fuzzy numbers</td>
</tr>
<tr>
<td>Preference ratio-based maximum operator approximation and its application in fuzzy flow shop scheduling</td>
<td>Sadi Nezhad, S. and Ghaleh Assadi, R. [20]</td>
<td>2008</td>
<td>This paper introduced an appropriate approximation for the maximum operator in which the weak dominance fuzzy numbers are ranked based on the concept of preference ratio</td>
<td>Triangular fuzzy numbers</td>
</tr>
<tr>
<td>Paper title</td>
<td>Author(s)</td>
<td>Year</td>
<td>Method for ranking</td>
<td>Notes</td>
</tr>
<tr>
<td>----------------------------------------------------------------------------</td>
<td>-----------------------------------------------</td>
<td>------</td>
<td>------------------------------------------------------------------------------------</td>
<td>------------------------------------------</td>
</tr>
<tr>
<td>Ranking of intuitionistic fuzzy numbers</td>
<td>Nayagam, V.I.G., Venkateshwar, G., and Sriram, G. [27]</td>
<td>2008</td>
<td>In this paper, a new method of intuitionistic fuzzy scoring to intuitionistic fuzzy numbers that generalizes Chen and Hwang's scoring method for ranking of intuitionistic fuzzy numbers has been introduced and studied</td>
<td>Intuitionistic Fuzzy Numbers</td>
</tr>
<tr>
<td>Similarity measure between generalized fuzzy numbers using quadratic-mean operator</td>
<td>Chen, S.J. [28]</td>
<td>2008</td>
<td>This paper presents a novel similarity measure that is based on the quadratic-mean operator to solve similarity measurement problems that involve generalized fuzzy numbers</td>
<td>Generalized fuzzy Numbers</td>
</tr>
<tr>
<td>The revised method of ranking fuzzy numbers with an area between the centroid and original points</td>
<td>Wang, Y.J. and Lee, H.S. [29]</td>
<td>2008</td>
<td>The paper proposed the revised method of ranking fuzzy numbers with an area between the centroid and original points</td>
<td></td>
</tr>
<tr>
<td>Trapezoidal approximations of fuzzy numbers preserving the expected interval</td>
<td>Grzegorzewski, P. [30]</td>
<td>2008</td>
<td>The Algorithms for calculating the proper approximations are proposed and some properties of the approximation operators are discussed</td>
<td></td>
</tr>
<tr>
<td>Ranking fuzzy numbers by distance minimization</td>
<td>Asady, A. and Zendelshem, A. [10]</td>
<td>2007</td>
<td>The authors proposed the use of the nearest point of support function for ranking FNs</td>
<td></td>
</tr>
<tr>
<td>Distance and similarity measures for fuzzy operators</td>
<td>Balopoulos, V., Hatimichalidis, A.G. and Papadopoulos, B.K. [31]</td>
<td>2007</td>
<td>This paper suggests a new family of normalized distance measures between fuzzy sets, based on binary operators and matrices.</td>
<td></td>
</tr>
<tr>
<td>A new approach for ranking fuzzy numbers based on fuzzy simulation analysis method</td>
<td>Sun, H. and Wu, J. [33]</td>
<td>2006</td>
<td>A combination of methods including computer and math application is developed</td>
<td></td>
</tr>
<tr>
<td>A new similarity measure of generalized fuzzy numbers based on geometric-mean averaging operator</td>
<td>Chen, S.J. [34]</td>
<td>2006</td>
<td>This paper presented a new similarity measure based on the geometric-mean averaging operator to handle the similarity measure problems of generalized fuzzy numbers</td>
<td>Generalized fuzzy numbers</td>
</tr>
<tr>
<td>A theoretical development on a fuzzy distance measure for fuzzy numbers</td>
<td>Chakraborty, C. and Chakraborty, D. [35]</td>
<td>2006</td>
<td>This paper introduced a fuzzy distance measure for generalized fuzzy numbers</td>
<td></td>
</tr>
<tr>
<td>On the centroids of fuzzy numbers</td>
<td>Wang, Y.M., Yang, J.B., Xu, D.L., and Chin, K.S. [36]</td>
<td>2006</td>
<td>This paper presented the correct centroid formulae for fuzzy numbers and justified them from the viewpoint of analytical geometry</td>
<td></td>
</tr>
<tr>
<td>The nearest trapezoidal form of a generalized left right fuzzy number</td>
<td>Abbasbandy, S. and Amirfakhrian, M. [37]</td>
<td>2006</td>
<td>This paper proposed a new approach to assigning distance between fuzzy numbers</td>
<td>Generalized LR fuzzy number</td>
</tr>
<tr>
<td>Selecting IS personnel use fuzzy GDSS based on metric distance method</td>
<td>Chen, L.S. and Ching., H.C. [38]</td>
<td>2005</td>
<td>This paper proposed a new approach to rank fuzzy numbers by metric distance</td>
<td></td>
</tr>
</tbody>
</table>
Table 1. An itemized list of different methods proposed for ranking the FNs (continued).

<table>
<thead>
<tr>
<th>Paper title</th>
<th>Author(s)</th>
<th>Year</th>
<th>Method for ranking</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ranking fuzzy numbers with preference weighting function expectations</td>
<td>Liu, X.W. and Han, S.L. [30]</td>
<td>2005</td>
<td>This paper extends the centroid expectation approach and proposes a preference weighting function expectation method to rank FNs</td>
<td></td>
</tr>
<tr>
<td>The nearest trapezoidal fuzzy number to a fuzzy quantity</td>
<td>Abbassianly, S. and Asady, B. [40]</td>
<td>2004</td>
<td>This paper introduced a fuzzy trapezoidal approximation using the metric (distance) between two fuzzy numbers</td>
<td>Trapexoidal fuzzy number</td>
</tr>
<tr>
<td>A new method for handling multi-criteria fuzzy decision-making problems using FN-IOWA operators</td>
<td>Chen, S.J. and Chen, S.M. [41]</td>
<td>2003</td>
<td>The method used fuzzy numbers to extend the traditional induced ordered weight averaging (IOWA) operator to present the fuzzy-number IOWA (FN-IOWA) operator, wherein fuzzy numbers are used to describe the argument values and the weights of the FN-IOWA operator, and the aggregation results are obtained by using fuzzy-number arithmetic operations</td>
<td>This method presented a new method for ranking fuzzy numbers</td>
</tr>
<tr>
<td>Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers</td>
<td>Chen, S.Y. and Chen, S.M. [42]</td>
<td>2003</td>
<td>This paper represented a method called the simple center of gravity method (SCGM) to calculate the center-of-gravity (COG) point of generalized fuzzy numbers. Then, it used the SCGM to propose a new method to measure the degree of similarity between generalized fuzzy numbers</td>
<td>Generalized fuzzy numbers</td>
</tr>
<tr>
<td>Ranking fuzzy numbers with an area between the centroid point and original point</td>
<td>Chu, T.C., Tsao, C.T. [7]</td>
<td>2002</td>
<td>The authors proposed their new approach for ranking FN that considered the area between the centroid and original points</td>
<td></td>
</tr>
<tr>
<td>An approximate approach for ranking fuzzy numbers based on left and right dominance</td>
<td>Chen, L.H. and Lu, H.W. [43]</td>
<td>2001</td>
<td>The proposed approach only requires a few left and right spreads at some n-levels of fuzzy numbers to determine the respective dominance of one fuzzy number over the other</td>
<td></td>
</tr>
<tr>
<td>Ranking fuzzy numbers by preference ratio</td>
<td>Motaries, M. and Sadeghnezhad, S. [44]</td>
<td>2001</td>
<td>In this method a preference function is denied by which fuzzy numbers are measured point by point and at each point the most preferred number is identified. Then, these numbers are ranked on the basis of their preference ratio</td>
<td>Triangular fuzzy numbers</td>
</tr>
<tr>
<td>Reasonable properties for the ordering of fuzzy quantities (I)</td>
<td>Wang, X. and Kerre, E.E. [45]</td>
<td>2001</td>
<td>A new method for ranking fuzzy numbers is proposed. The method considers the overall possibility distributions of fuzzy numbers in their evaluations for ranking FNs</td>
<td>This method evaluates fuzzy numbers with a satisfaction function and the viewpoint given by a user.</td>
</tr>
<tr>
<td>A method for ranking fuzzy numbers and its application to decision-making</td>
<td>Lee, Kwang, H. and Lee, J.H. [46]</td>
<td>1999</td>
<td>A modified geometrical distance method is presented to measure the distance between two fuzzy numbers</td>
<td>Triangular fuzzy numbers</td>
</tr>
<tr>
<td>A model and algorithm of fuzzy product positioning</td>
<td>Hsieh, C.H. and Chen, S.H. [47]</td>
<td>1999</td>
<td>A new method for ranking fuzzy numbers is proposed. The method considers the overall possibility distributions of fuzzy numbers in their evaluations for ranking FNs</td>
<td>This method evaluates fuzzy numbers with a satisfaction function and the viewpoint given by a user.</td>
</tr>
<tr>
<td>A new fuzzy arithmetic</td>
<td>Ma, M., Friedman, M. and Kandel, A. [48]</td>
<td>1999</td>
<td>This paper presented fuzzy numbers with a new parametric form. Based on this representation, a new fuzzy arithmetic is defined</td>
<td></td>
</tr>
</tbody>
</table>
Table 1. An itemized list of different methods proposed for ranking the FNs (continued).

<table>
<thead>
<tr>
<th>Paper title</th>
<th>Author(s)</th>
<th>Year</th>
<th>Method for ranking</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A simple fuzzy group decision making method</td>
<td>Cheng, C.H. [49]</td>
<td>1999</td>
<td>First the intuition ranking method, then the alpha cut method and finally the fuzzy mean and spread numbers are used for ranking</td>
<td>The defuzzification value of the trapezoidal fuzzy numbers is used for ranking FNs</td>
</tr>
</tbody>
</table>

On a canonical representation of fuzzy numbers


This paper used two parameters, value and ambiguity parameters to obtain canonical representations and to deal with fuzzy numbers in decision-making problems.

Some remarks on distances between fuzzy numbers


A fuzzy distance between fuzzy numbers is introduced and its basic properties are studied.

A new approach for ranking fuzzy numbers by distance method


This paper indicated that Liou and Wang’s (1992) [53] method has a defect in ranking normal and provided a method improvement.

Ranking fuzzy numbers with integral value


This paper used the concepts of the integral values for ranking normal and non-normal FNs.

A new index for ranking fuzzy numbers


A procedure for ranking discrete fuzzy numbers is presented, proposed in F. Choobineh and Haibash, Li (1990).

Figure 1. The left and right deviation degrees of $A_i$.

$$d^L_i = \int_{x_{min}}^{a_i} \left( \mu_{A_{min}}(x) - \mu_{A_i}(x) \right) dx, \quad (6)$$

and:

$$d^R_i = \int_{b_i}^{x_{max}} \left( \mu_{A_{max}}(x) - \mu_{A_i}(x) \right) dx, \quad (7)$$

where $a_i$ is the abscissa of the crossover point of $\mu_{A_i}$ and $\mu_{A_{min}}(x)$ and $b_i$ is that of $\mu_{A_{max}}(x)$ and $\mu_{A_i}(x)$, $i = 1, 2, \ldots, n$.

Wang et al. [17] considered an index for ranking by using these deviation degrees. They calculated the expectation value of centroid of each FN and constructed the transfer coefficient of each FN $A_i(i \in 1, 2, \ldots, n)$ as follows:

$$\lambda_i = \frac{M_i - M_{\text{min}}}{M_{\text{max}} - M_{\text{min}}} \quad (8)$$

where $M_i$ is the expectation value of centroid of L-R fuzzy number $A_i = (m_i, n_i, \alpha_i, \beta_i)$ defined as:

$$M_i = \frac{\int_{m_{n_i} - 0}^{n_i + n_i} x \mu_{A_i}(x) dx}{\int_{m_{n_i} - 0}^{n_i + n_i} \mu_{A_i}(x) dx} \quad (9)$$

and $M_{\text{min}} = \min(m_1, m_2, \ldots, m_n)$ and $M_{\text{max}} = \max(m_1, m_2, \ldots, m_n)$ and $M_{\text{max}} \neq M_{\text{min}}$.

Ultimately the ranking measure is given by:

$$d_i = \begin{cases} \frac{\lambda_i d^L_i}{1 + (1 - \lambda_i) d^R_i} & M_{\text{max}} \neq M_{\text{min}} \quad i = 1, 2, \ldots, n \\ \frac{d^L_i}{1 + d^R_i} & M_{\text{max}} = M_{\text{min}} \quad i = 1, 2, \ldots, n \end{cases} \quad (10)$$

where the greater the $d_i$ is, the larger is the FN $A_i$.

3.2. Area ranking of fuzzy numbers based on positive and negative ideal points

Wang and Luo [18] considered a positive ideal point and a negative ideal point. For a set of FNs $A_1, A_2, \ldots, A_n$ they defined the positive and negative ideal points as $x_{\text{max}} = \sup S$ and $x_{\text{min}} = \inf S$, where $S = \bigcup_{i=1}^{n} s_i$ and $s_i$ is the support set of the $A_i$. Let $A_i = (a_i, b_i, c_i, d_i)$ be one of the FNs to be compared whose membership function is defined by Eq. (1). The areas $S_L(i)$ and
3.3. Analysis of the shortage

The main disadvantage in both of these methods is that all of FNs, having infimum equal to \( x_{\min} \) and left spread equal to zero, are considered equal, but this is not true. We will explain this in the following.

Consider Wang et al.’s [17] method. In this method for every L-R fuzzy number \( A_i = (m_i, n_i, \alpha_i, \beta_i) \), \( a_i \) the abscissa of the crossover point of \( \mu_{A_l}(x) \) and \( \mu_{A_u}(x) \), is equal to:

\[
A_i = \frac{x_{\max}(m_i - \alpha_i) + m_i(1 - x_{\min}) - \alpha_i}{x_{\max} - x_{\min} + 1 - \alpha_i}, \tag{20}
\]

if \( m_i = x_{\min} \) and \( \alpha_i = 0 \) then:

\[
A_i = \frac{x_{\max}(x_{\min}) + m_i(1 - x_{\min}) - \alpha_i}{x_{\max} - x_{\min} + 1} = x_{\min},
\]

so:

\[
d_i = \int_{x_{\min}}^{x_{\max}} (\mu_{A_u}(x) - \mu_{A_l}(x)) \, dx = 0.
\]

Therefore, by Eq. (10), for every couple of this described FN, we reach to:

\[
d_i = 0.
\]

Also in Wang and Luo’s [18] method for a set of FNs \( A_1, A_2, \ldots, A_n \), \( A_i = (a_i, b_i, c_i, d_i) \), if \( a_i = x_{\min} \) \((x_{\min} = \inf S, \) where \( S = \bigcup_{i=1}^{n} s_i \) and \( s_i \) is the support set of the \( A_i \)) and the left spread is equal to zero \((b_i = a_i)\), then:

\[
S_i = \int_{x_{\min}}^{x_{\max}} dx + \int_{x_{\min}}^{x_{\max}} (1 - f_{A_l}(x)) \, dx = 0.
\]

and so:

\[
\text{RIA}_2(i) = \frac{S_i}{S_l(i)r_l(i) + S_R(i)r_R'(i)},
\]

and therefore, all of these FNs are considered equal.

Therefore, in both of these methods, all of the fuzzy numbers, having the infimum equal to \( x_{\min} \) and the left spread equal to zero, are considered equal. For example consider L-R fuzzy numbers \( A = (2, 0, 0), B = (2, 0, 3), C = (2, 0, 6) \) and \( D = (2, 5, 0, 3) \) as shown in Figure 3. By Wang et al.’s [17] method as \( d_A^L = d_B^L = d_C^L = d_D^L = 0 \) so \( d_A = d_B = d_C = d_D = 0 \). Therefore, using this method results in \( A = B = C = D \). Also in Wang and Luo’s [18] method, as \( S_l(A) = S_l(B) = S_l(C) = S_l(D) = 0 \), using RIA_2 results in \( A = B = C = D \). However, Figure 3, intuitively, shows that \( A < B < C < D \).
4. Improvement in the area ranking and deviation degree methods by considering decision maker’s risk attitudes to ranking

In this section an improved approach is proposed to overcome the shortage of Wang et al. [17] and Wang and Luo [18] methods. For a set of FNs $A_1, A_2, \ldots, A_n$, consider $x_{\text{max}} = \sup S$ and $x_{\text{min}} = \inf S$, where $S = \bigcup_{i=1}^{n} s_i$ and $s_i$ is the support set of the $A_i$. The proposed method is based on considering two areas. One area is under minimization set from $x_{\text{min}}$ to the crossover point of minimization set and $f(x)$. Another is the area under maximization set between two points $a_i$ and $x_{\text{max}}$ (see Figure 4). However, to take into account the decision maker’s (DM’s) risk attitude, the number $0 \leq \varepsilon \leq 1$ is considered to show the DM’s attitude toward risks, so that the greater is the $\varepsilon$, the more is the DM’s attitude toward the risk, and vice versa. To consider the DM’s risk attitude into ranking the $\varepsilon$ is engaged in the maximization and minimization sets by shifting the minimization set as $\varepsilon$ to the left and the maximization set as $1 - \varepsilon$ to the right. This makes the first and second areas, as mentioned above, to be larger and smaller, respectively. The areas are shown in Figure 4. By engaging the number $0 \leq \varepsilon \leq 1$ in the minimization and the maximization sets, the two new sets are obtained which are named by $t_{\text{max}_\varepsilon}$ and $t_{\text{min}_\varepsilon}$, respectively, and their membership functions are:

$$
\mu_{t_{\text{max}_\varepsilon}}(x) = \begin{cases} 
\frac{x_{\text{max}} - x_{\text{min}} - \varepsilon}{x_{\text{max}} - x_{\text{min}} + 1 - \varepsilon} & x_{\text{min}} - \varepsilon \leq x \leq x_{\text{max}} + 1 - \varepsilon \\
0 & \text{else}
\end{cases}
$$

(21)

and $0 \leq \varepsilon \leq 1$.

It is obvious that for every fuzzy number $A$, if $A$ is larger, the area $d_L$ is larger and the area $d_R$ is smaller.

For a trapezoidal fuzzy number $A = (a, b, c, d)$, $a_t$ and $b_t$, the crossover points, are:

$$
a_t = \frac{a(x_{\text{max}} + 1 - \varepsilon) - b(x_{\text{min}} - \varepsilon)}{x_{\text{max}} - x_{\text{min}} + a - b + 1},
$$

(23)

$$
b_t = \frac{c(x_{\text{max}} + 1 + \varepsilon) - d(x_{\text{min}} + \varepsilon)}{x_{\text{max}} - x_{\text{min}} + c - d + 1},
$$

(24)

For a triangular fuzzy number $A = (a, b, c)$, $a_t$ is the same as Eq. (23) and $b_t$ is easily obtained by replacing $c$ and $d$ by $b$ and $c$, in Eq. (24).

The areas $d_L$ and $d_R$ are computed as follows:

$$
d_L = \int_{x_{\text{min}}}^{a_t} \mu_{t_{\text{max}_\varepsilon}}(x)dx,
$$

(25)

$$
d_R = \int_{b_t}^{x_{\text{max}}_{\text{min}_\varepsilon}} \mu_{t_{\text{max}_\varepsilon}}(x)dx.
$$

(26)

It is obtained that for $0 \leq \varepsilon \leq 1$, $d_L$ and $d_R$ are monotonically increasing and decreasing functions of $\varepsilon$, respectively. For a trapezoidal fuzzy number $A = (a, b, c, d)$, the ranking measure value($A$) is defined as:

$$
\text{Value}(A) = \frac{a + d + d_L}{1 + d_R},
$$

(27)

and for a triangular FN $A = (a, b, c)$, an area ranking measure is defined as:

$$
\text{Value}(A) = \frac{a + c + d_L}{1 + d_R},
$$

(28)

where the larger is Value($A$), the greater the fuzzy number $A$ is.

5. Numerical examples

In this section, we represent numerical examples to verify the validity of the proposed approach. The results are compared with some other methods.

**Example 1.** Consider triangular fuzzy numbers $A = (2, 2, 2)$, $B = (2, 2, 8)$, and trapezoidal fuzzy number $C = (2, 2, 3, 4)$ in Figure 5. From Figure 5 it is obvious that $A < C < B$, but in Wang and Luo’s [18] method, by regarding to measure RIA2 and also by using Wang et al.’s [17] method, the results will be $A = B = C$. 
Our result for ranking these FNs is shown in Figure 6 and for every $0 \leq \varepsilon \leq 1$ we see that $A \succ B \succ C$.

Also the inverse and symmetry of these FNs are $A^{-1} = (0.5, 0.5, 0.5)$, $B^{-1} = (0.125, 0.5, 0.5, 0.5)$, $C^{-1} = (0.25, 0.33, 0.5, 0.5)$, and $(\neg A) = (-2, -2, -2, -2)$, $(\neg B) = (-8, -2, -2, -2)$, $(\neg C) = (-4, -3, -2, -2)$. Our approach’s results for inverse and symmetry of FNs are shown in Figures 7 and 8 and in both of them we reach to ranking $A^{-1} \succ B^{-1} \succ C^{-1}$ and $(\neg A) \succ (\neg C) \succ (\neg B)$ which are true.

**Example 2.** Consider two triangular fuzzy numbers $A = (2, 3, 3)$ and $B = (2, 2, 5)$, as show in Figure 9. In Wang and Luo’s [18] method, by regarding the indice RIA$_2$, the ranking is $A \succ B$, and by noting measure RIA$_1$ for $\alpha = 0.5$ the result is $A \sim B$. Also Wang et al.’s [17] method results $A \succ B$. All of these rankings are false, but our approach’s result that is shown in Figure 10, for every $0 \leq \varepsilon \leq 1$ reaches to $B \succ A$ that is true.

**Example 3.** Consider FNs $A = (2.5, 3.3, 5)$ and $B = (0.3, 6)$ shown in Figure 11. It is obvious that the decision maker prefers the result $A \succ B$ because it obtains more precise information, but by Wang and Luo’s [18] approach the different rankings are obtained for the different values of $\alpha$. However our approach for every $0 \leq \varepsilon \leq 1$ reaches to $A \succ B$. The results for our method and Wang and Luo’s [18] methods are summarized in Table 2. Also Table 3 summarizes the results obtained by some other methods.
Table 2. Ranking results of FNs A, B in Example 3 for our method and Wang and Luo’s [18] method.

<table>
<thead>
<tr>
<th>Proposed approach</th>
<th>Wang and Luo’s [18] approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RIA1</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>0</td>
<td>2.778685</td>
</tr>
<tr>
<td>0.5</td>
<td>2.52708</td>
</tr>
<tr>
<td>1</td>
<td>2.340817</td>
</tr>
</tbody>
</table>

Table 3. Ranking results of FNs A, B in Example 3 by different approaches.

<table>
<thead>
<tr>
<th>Ranking approach</th>
<th>A</th>
<th>B</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deviation degree</td>
<td>0.5</td>
<td>0.89256288</td>
<td>A ( \succ ) B</td>
</tr>
<tr>
<td>Sign distance (p = 1)</td>
<td>6</td>
<td>6</td>
<td>A ( \succ ) B</td>
</tr>
<tr>
<td>Sign distance (p = 2)</td>
<td>4.32</td>
<td>4.8989</td>
<td>B ( \succ ) A</td>
</tr>
<tr>
<td>Chu and Tsao</td>
<td>1.5</td>
<td>1.5</td>
<td>A ( \succ ) B</td>
</tr>
<tr>
<td>Cheng’s distance index</td>
<td>3.041</td>
<td>3.041</td>
<td>A ( \succ ) B</td>
</tr>
<tr>
<td>Cheng’s CV index</td>
<td>0.00833</td>
<td>0.3</td>
<td>A ( \succ ) B</td>
</tr>
<tr>
<td>Xu and Zhai [15]</td>
<td>( C(A) = 3 ), ( D(A) = 18.33 ), ( C(B) = 3 ), ( D(B) = 24 )</td>
<td>A ( \succ ) B</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Proposed method’s result for FNs in Example 4.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.875</td>
<td>9.2137</td>
<td>9.4167</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>6.647</td>
<td>8.1839</td>
<td>8.3687</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>5.961</td>
<td>7.3091</td>
<td>7.4545</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
</tbody>
</table>

Figure 12. FNs in Example 4.

Example 4. Consider L-R FNs in Ref. [18], i.e. \( A = (6,6,1,1)_{LR} \), \( B = (6,6,0,1,1)_{LR} \) and \( C = (6,6,0,1)_{LR} \) as shown in Figure 12. Table 4 shows the results obtained by our approach.

Table 5 gives the ranking results obtained by some other methods. Also our approach’s results for symmetry of these FNs are given in Table 6.

Table 5. Ranking results of L-R FNs A, B, C in Example 4 by different approaches.

<table>
<thead>
<tr>
<th>Ranking approach</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deviation degree</td>
<td>0.25</td>
<td>0.5339</td>
<td>0.5625</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>Sign distance (p = 1)</td>
<td>6.12</td>
<td>12.45</td>
<td>12.5</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>Sign distance (p = 2)</td>
<td>8.52</td>
<td>8.82</td>
<td>8.85</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>Chu and Tsao</td>
<td>3</td>
<td>3.126</td>
<td>3.085</td>
<td>A ( \succ ) C ( \prec ) B</td>
</tr>
<tr>
<td>Cheng’s distance index</td>
<td>6.021</td>
<td>6.349</td>
<td>6.7519</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
<tr>
<td>Cheng’s CV index</td>
<td>0.028</td>
<td>0.0098</td>
<td>0.0089</td>
<td>C ( \succ ) B ( \prec ) A</td>
</tr>
<tr>
<td>Xu and Zhai [15]</td>
<td>6</td>
<td>6.225</td>
<td>6.25</td>
<td>A ( \succ ) B ( \prec ) C</td>
</tr>
</tbody>
</table>

6. Conclusion

This paper introduced an improved approach to overcome the shortage of area ranking and deviation degree methods in ranking different fuzzy numbers. Considering decision maker’s risk attitudes is an important point in ranking FNs that was considered in this paper. Whereas a number of approaches fail to rank the symmetry and inverse of FNs, this approach efficiently acts with these cases. The examples given in this paper shows that the proposed approach can
efficiently rank different FNs compared to other approaches.

Acknowledgment

The authors would like to thank the anonymous reviewers whose suggestions helped this paper be improved.

References


Biographies

Reza Ghasemi received his BS degree in Industrial Engineering from Tafresh University of Tafresh, Iran (2010), and now is an MS student of Industrial Engineering in Khaje Nasiroddin Toosi University (KNTU) of technology of Tehran, Iran. Recently he has submitted a Technical Note on shortage of two fuzzy number ranking methods. He also has done researches on lean production and ways to reduce wastes in healthcare systems. His research interests are: Fuzzy sets and applications, Bayesian inferences, statistical quality control and simulation.
Mohsen Nikfar is an MS student of industrial engineering in Khaje Nasiroddin Toosi University (KNTU) of technology of Tehran, Iran. He received his BS degree from Hormozgan University of Hormozgan, Iran, in Industrial engineering (2007). He cooperated with Iranian association of productivity in 2005 for planning productivity document of Shahid Sadoghi University of Medical Sciences. His areas of research include: scheduling, fuzzy sets and SCM.

Emad Roghanian is an assistant professor and a faculty member of the department of Industrial Engineering at KN Toosi University of Technology in Tehran, Iran. He received his bachelor degree from Isfahan University of Technology and his master and PhD degrees from Iran University. His fields of interests are SCM and Logistic, project management, fuzzy logic and fuzzy methods, stochastic and performance measurement.