A new technique for non-linear two-dimensional wave equations

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Abstract In this paper, a new technique, namely, the New Homotopy Perturbation Method (NHPM) is applied for solving a non-linear two-dimensional wave equation. The two most important steps in application of the new homotopy perturbation method are to construct a suitable homotopy equation and to choose a suitable initial guess. Comparison between our solution with Homotopy perturbation method (HPM) shows that the NHPM is effective and accurate in solving these kinds of equation.

1. Introduction

Since the governing equations in many experiments in engineering as well as in the sciences, lead to the wave equation, this equation has attracted much attention, and its solution is one of the most interesting tasks undertaken by mathematicians. Analytic methods commonly used for solving the wave equation are very restricted and can be used in special cases, so they cannot be used to solve equations of numerous realistic scenarios. Numerical techniques, which are usually used, encounter difficulties in terms of the size of computational tasks needed and usually the round-off error causes a loss of accuracy. One of the most powerful methods to approximately solve nonlinear differential equations is the homotopy perturbation method [1,2]. The HPM is based on the use of a power series, which transforms the original nonlinear differential equation into a series of linear differential equations. This method has been widely used for solving various functional equations, arising in real world modeling. For example, the inverse heat conduction problem [3], elasto-plastic analysis [4], the three-dimensional problem of condensation film [5], thin film flow [6], partial differential equations of fractional order in finite domains [7], nonlinear vibration [8], and many others [9–22]. In this paper a modified version of HPM, called the new homotopy perturbation method (NHPM) has been introduced for solving the non-linear two-dimensional wave equation, and it is shown that the new technique performs much better than the HPM.

2. Basic ideas of the NHPM

Consider the following two-dimensional non-linear differential equation, subject to the following initial conditions:

\[ \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 u}{\partial t^2} = \phi(x, t), \quad 0 \leq x, t \leq 1, \]  
\[ u(0, t) = f(t), \quad \frac{\partial}{\partial x} u(0, t) = g(t). \]

To solve Eq. (1), let us construct the following homotopy:

\[ (1 - p) \left( \frac{\partial^2 U}{\partial x^2} (x, t) - u_0(x, t) \right) \]

\[ + p \left( \frac{\partial^2 U}{\partial x^2} (x, t) - \frac{\partial^2 U}{\partial t^2} (x, t) - \phi(x, t) \right) = 0. \]

Or:

\[ \frac{\partial^2 U}{\partial x^2} (x, t) = u_0(x, t) \]

\[ - p \left( u_0(x, t) - \frac{\partial^2 U}{\partial t^2} (x, t) - \phi(x, t) \right). \]
Applying the inverse operator, \( L^{-1} = \int_0^x \int_0^x (\ ) dx \), to both sides of Eq. (2), we obtain

\[
U(x, t) = U(0, t) + xU_t(0, t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx
\]

\[-p \int_0^x \int_0^x \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) - \phi(x, t) \right) \, dx \, dx,
\]

where \( U(0, t) = u(0, t) \), and \( U_t(0, t) = u_t(0, t) \).

Suppose the solution of Eq. (3) as the summation of a series say

\[
U = U_0 + pU_1 + p^2U_2 + \cdots
\]

where \( U_i \) are unknown functions which should be determined.

The difference between the two methods (HPM and NHPM) starts from the form of initial approximation of the solution. Now, suppose that the initial approximation to the solution \( u_0(x, t) \), is in the form:

\[
u_0(x, t) = \sum_{n=0}^{\infty} a_n(t)P_n(x),
\]

where \( a_n(t), a_1(t), a_2(t), \ldots \) are unknown coefficients and \( P_n(x), P_1(x), P_2(x), \ldots \) are specified functions depending on the problem. Substituting (4) and (5) into Eq. (3), classifying the terms with identical powers of \( p \), and equating coefficients of \( p^i \) to zero, results in:

\[
p^0 : U_0(x, t) = f(t) + xg(t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx.
\]

\[
p^1 : U_1(x, t) = -\int_0^x \int_0^x \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) - \phi(x, t) \right) \, dx \, dx.
\]

\[
p^2 : U_2(x, t) = -\int_0^x \int_0^x \left( -U \frac{\partial^2 U}{\partial t^2}(x, t) - U_1 \frac{\partial^2 U}{\partial t^2}(x, t) \right) \, dx \, dx,
\]

\[
p^{i+1} : U_{i+1}(x, t) = -\int_0^x \int_0^x \left( \sum_{k=0}^{i} U_k \frac{\partial^2 U_{i-k}}{\partial t^2}(x, t) \right) \, dx \, dx.
\]

Considering the hypothesis, \( U_1(x, t) = 0 \), then, results in:

\[
U_2(x, t) = U_3(x, t) = \cdots = 0.
\]

Therefore the exact solution would be obtained as the following:

\[
u(x, t) = u_0(x, t).
\]

It is worthwhile to note that if \( \phi(x, t) \), and \( u_0(x, t) \) are analytic at \( x = x_0 \), then their Taylor series are defined as:

\[
u_0(x, t) = \sum_{n=0}^{\infty} a_n(t)(x - x_0)^n.
\]

\[
\phi(x, t) = \sum_{n=0}^{\infty} a_n^*(t)(x - x_0)^n.
\]

which can be used in Eq. (6), where \( a_1(t), a_2(t), a_3(t), \ldots \) are unknown coefficients that must be determined and \( a_1^*(t), a_2^*(t), a_3^*(t), \ldots \) are known ones.

To illustrate the effectiveness of the method, NHPM will be applied to some equations, as examples.

**Example 1.** Consider the following two-dimensional non-linear wave equation:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = -t^2 \sin x - 2t^2 \sin^2 x \quad 0 \leq x, t \leq 1,
\]

with the initial conditions:

\[
u(0, t) = 0, \quad \frac{\partial}{\partial x}u(0, t) = t^2.
\]

With the exact solution is:

\[
u(x, t) = t^2 \sin x.
\]

**Homotopy Perturbation Method:**

Using HPM, leads to:

\[
u_0 = xt^2,
\]

\[
u_1 = \frac{1}{6} t^2 x^4 + \frac{3}{4} t^2 - \frac{3}{4} t^2 \cos^2 x - \frac{3}{4} t^2 x^2,
\]

\[
u_2 = \frac{1}{63} t^2 x^6 + \frac{1}{4} t^2 x^4 + \frac{3}{4} t^2 x^2 \cos^2 x - \frac{3}{4} t^2 \sin x \cos x \cdot \cdots.
\]

Therefore the approximate solution of Example 1 by HPM can be written as:

\[
u = xt^2 + \frac{1}{6} t^2 x^4 + \frac{3}{4} t^2 - \frac{3}{20} t^2 x^2 - \frac{3}{4} t^2 x^2 \frac{1}{63} t^2 x^2 + \frac{1}{4} t^2 x^3 - \frac{3}{4} t^2 \cos^2 x + \frac{3}{4} t^2 x^2 \cos^2 x - \frac{3}{4} t^2 \sin x \cos x \cdot \cdots.
\]

To solve Eq. (7), by the NHPM, we construct the following homotopy:

\[
\frac{\partial^2 U(x, t)}{\partial x^2}(x, t) = u_0(x, t)
\]

\[-p \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) + t^2 \sin x + 2t^2 \sin^2 x \right).
\]

By integration of Eq. (8) we have:

\[
u(x, t) = U(0, t) + xU_t(0, t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx
\]

\[-p \int_0^x \int_0^x \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) + t^2 \sin x + 2t^2 \sin^2 x \right) \, dx \, dx.
\]

Suppose that the solution of Eq. (9) is in the form (4). Substituting (4) into Eq. (9), and equating the terms with identical powers of \( p \), results in:

\[
p^0 : U_0(x, t) = U(0, t) + xU_t(0, t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx,
\]

\[
p^1 : U_1(x, t) = \int_0^x \int_0^x \left( U_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) + t^2 \sin x + 2t^2 \sin^2 x \right) \, dx \, dx,
\]
\[ p^2 : U_2(x, t) = - \int_0^x \int_0^x \left( -u_0 \frac{\partial^2 U_1}{\partial t^2}(x, t) - U_1 \frac{\partial^2 U_0}{\partial t^2}(x, t) \right) \, dx \, dx, \]
\[ \vdots \]
\[ p^{i+1} : U_{i+1}(x, t) = - \int_0^x \int_0^x \left( \sum_{k=0}^{i} \int_0^x \frac{\partial^2 U_{i-k}}{\partial t^2}(x, t) \right) \, dx \, dx, \]
\[ \vdots \]

Assuming \( u_0(x, t) = \sum_{n=0}^{\infty} a_n(t)x^n \), \( U(0, t) = u(0, t) \), \( U_i(0, t) = u_i(0, t) \) and \( U_i(x, t) = 0 \), then we have:

\[ U_i(x, t) = \left( -\frac{1}{2} a_0(t) \right) x^2 + \left( -\frac{1}{6} a_1(t) - \frac{1}{6} t^2 \right) x^3 \]
\[ + \left( -\frac{1}{12} a_2(t) \right) x^4 + \left( -\frac{1}{20} a_3(t) + \frac{1}{40} t^2 \right) x^5 \]
\[ + \frac{1}{20} a_4(t) - \frac{1}{120} t^2 \right) x^6 + \cdots \]

It can be easily shown that:

\[ a_0(t) = 0, \quad a_1(t) = -t^2, \quad a_2(t) = 0, \]
\[ a_3(t) = \frac{1}{6} t^2, \ldots \]

This implies that:

\[ u(x, t) = U_0(x, t) = \frac{t^2}{2} + a_0(t) \frac{x^2}{2} + \frac{1}{6} a_1(t) x^3 \]
\[ + \frac{1}{12} a_2(t) x^4 + \frac{1}{20} a_3(t) x^5 + \cdots = t^2 \sin x. \]

The series has been recognized as an exact solution.

**Example 2.** Consider the following two-dimensional nonlinear wave equation [23]:

\[ \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 u}{\partial t^2} = 1 - \frac{x^2 + t^2}{2}, \quad 0 \leq x, t \leq 1, \] (10)

with initial conditions:

\[ u(0, t) = \frac{t^2}{2}, \quad \frac{\partial u}{\partial x}(0, t) = 0. \]

The exact solution of this equation is:

\[ u(x, t) = \frac{x^2 + t^2}{2}. \]

To solve Eq. (10), by the NHPM, we construct the following homotopy:

\[ \frac{\partial^2 U}{\partial x^2}(x, t)(x, t) = u_0(x, t) - p \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) \right) - 1 + \frac{x^2 + t^2}{2}. \]

(11)

Applying the inverse operator, \( L^{-1} = \int_0^x f(y) \, dy \) to both sides of this equation, we obtain:

\[ U(x, t) = U(0, t) + uU_0(0, t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx \]
\[ - p \int_0^x \int_0^x \left( u_0(x, t) - U \frac{\partial^2 U}{\partial t^2}(x, t) - 1 \right) \]
\[ + \frac{x^2 + t^2}{2} \]

(12)

Suppose the solution of Eq. (12) to have the following form:

\[ U = U_0 + puU_1 + p^2 U_2 + \cdots \]

where \( U_i \) are unknown functions which should be determined. Substituting Eq. (13) into Eq. (12), equating collection of the terms with identical powers of \( p \), and equating the coefficients of \( p^i \) leads to:

\[ p^0 : U_0(x, t) = U(0, t) + xU_1(0, t) + \int_0^x \int_0^x u_0(x, t) \, dx \, dx, \]
\[ p^1 : U_1(x, t) = - \frac{1}{2} a_0(t) x^2 \]
\[ + \left( -\frac{1}{2} a_2(t) \right) x^3 \]
\[ + \left( -\frac{1}{20} a_3(t) + \frac{1}{40} t^2 \right) x^4 \]
\[ + \frac{1}{20} a_4(t) - \frac{1}{120} t^2 \right) x^5 + \cdots \]

Assuming \( u_0(x, t) = \sum_{n=0}^{\infty} a_n(t)x^n \), \( P_k(x) = x^k \), \( U(0, t) = u(0, t) \), \( U_1(0, t) = u_1(0, t) \) and solving the above equation for \( U_1(x, t) \) leads to:

\[ U_1(x, t) = \left( -\frac{1}{2} a_0(t) + \frac{1}{2} \right) x^2 \]
\[ + \left( -\frac{1}{12} a_2(t) \right) x^3 \]
\[ + \left( -\frac{1}{20} a_3(t) + \frac{1}{40} t^2 \right) x^4 \]
\[ + \frac{1}{20} a_4(t) - \frac{1}{120} t^2 \right) x^5 + \cdots \]

Considering the hypothesis \( U_1(x, t) = 0 \), coefficients \( a_n(x)(n = 1, 2, 3, \ldots) \) will be determined as follows:

\[ a_0(t) = 1, \quad a_1(t) = 0, \quad a_2(t) = 0, \quad a_3(t) = 0, \ldots \]

Therefore, the solution of Eq. (10) becomes as:

\[ u(x, t) = u_0(x, t) = \frac{t^2}{2} + a_0(t) \frac{x^2}{2} + \frac{1}{6} a_1(t) x^3 \]
\[ + \frac{1}{12} a_2(t) x^4 + \frac{1}{20} a_3(t) x^5 + \cdots = \frac{x^2 + t^2}{2}. \]

**Example 3.** Consider the coupled system:

\[ \frac{\partial^2 u}{\partial x^2} - v \frac{\partial^2 u}{\partial t^2} = 2 - 2x^2 - 2t^2, \]
\[ \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 v}{\partial t^2} + u \frac{\partial^2 u}{\partial t^2} = 1 + \frac{3}{2} x^2 + \frac{3}{2} t^2. \]

(14)
Subject to initial conditions:
\[ u(0, t) = t^2, \quad \frac{\partial}{\partial x} u(0, t) = 0, \]
\[ v(0, t) = \frac{1}{2} t^2, \quad \frac{\partial}{\partial x} v(0, t) = 0. \]

Homotopy Perturbation Method:

Using HPM leads to:
\[ U_0 = t^2 + x^2 - \frac{1}{6} x^4, \]
\[ V_0 = \frac{1}{2} t^2 + \frac{1}{2} x^2 + \frac{3}{4} x^3 t^2 + \frac{1}{8} x^4, \]
\[ U_1 = -x^2 t^2 + \frac{1}{10} x^4 + \frac{7}{360} x^6 + \frac{1}{12} x^4 t^2 - \frac{1}{112} x^6 - \frac{1}{10} x^3 t^2, \]
\[ V_1 = \frac{3}{4} x^4 t^2 - \frac{1}{8} x^4 + \frac{77}{720} x^6 + \frac{11}{24} x^4 t^2 - \frac{1}{384} - \frac{7}{240} x^2 t^2, \]

The following approximations to the solution are considered as the following:
\[ u = t^2 + x^2 - \frac{1}{2} x^2 + \ldots \]
\[ v = \frac{1}{2} t^2 + \frac{1}{2} x^2 + \frac{3}{4} x^3 t^2 + \frac{1}{8} x^4 + \ldots. \]

To solve system (14), by the NHPM, we construct the following homotopies:
\[ \frac{\partial^2 U}{\partial x^2} = u_0 - p \left( u_0 - V \frac{\partial^2 U}{\partial t^2} - U \frac{\partial^2 V}{\partial t^2} - 2 + 2 x^2 + 2 t^2 \right), \]
\[ \frac{\partial^2 V}{\partial x^2} = v_0 - p \left( v_0 - \frac{\partial^2 V}{\partial t^2} + U \frac{\partial^2 U}{\partial t^2} - 1 - \frac{3}{2} x^2 - \frac{3}{2} t^2 \right). \] (15)

Applying the inverse operator, \( L^{-1} = \int_0^x \int_0^t \)dx to both sides of the above equations, we obtain:
\[ U = U(0, t) + x U_x(0, t) + \int_0^x \int_0^t u_0 dx dx - p \]
\[ \times \int_0^x \int_0^t \left( u_0 - V \frac{\partial^2 U}{\partial t^2} - U \frac{\partial^2 V}{\partial t^2} - 2 + 2 x^2 + 2 t^2 \right) dx dx, \]
\[ V = V(0, t) + x V_x(0, t) + \int_0^x \int_0^t v_0 dx dx - p \]
\[ \times \int_0^x \int_0^t \left( v_0 - \frac{\partial^2 V}{\partial t^2} + U \frac{\partial^2 U}{\partial t^2} - 1 - \frac{3}{2} x^2 - \frac{3}{2} t^2 \right) dx dx. \] (16)

Suppose the solutions of system (16) are, as assumed in (4). Substituting Eqs. (4) into Eqs. (16), collecting the terms with the same powers of \( p \), and equating each coefficient of all powers of \( p \) to zero, leads to:
\[ p^0 : \]
\[ U_0(x, t) = U(0, t) + x U_x(0, t) + \int_0^x \int_0^t u_0(x, t) dx dx, \]
\[ V_0(x, t) = V(0, t) + x V_x(0, t) + \int_0^x \int_0^t v_0(x, t) dx dx, \]
\[ U_1(x, t) = \int_0^x \int_0^t \left( -u_0 + V_0 \frac{\partial^2 U_0}{\partial t^2} + U_0 \frac{\partial^2 V_0}{\partial t^2} - 2 + 2 x^2 + 2 t^2 \right) dx dx, \]
\[ V_1(x, t) = \int_0^x \int_0^t \left( -v_0 + V_0 \frac{\partial^2 V_0}{\partial t^2} - 1 - \frac{3}{2} x^2 - \frac{3}{2} t^2 \right) dx dx. \]

By taking each of the Taylor series of \( U_1(x, t) \), \( V_1(x, t) \) equal to zero, we have:
\[ U_1(x, t) = \left( -\frac{1}{2} a_0(t) + 1 \right) x^2 + \left( -\frac{1}{6} a_1(t) \right) x^3 \]
\[ + \left( -\frac{1}{12} a_2(t) + \frac{1}{12} b_0(t) + \frac{1}{24} a_3(t) \right) x^4 + \ldots = 0, \]
\[ V_1(x, t) = \left( -\frac{1}{2} b_0(t) + \frac{1}{2} + \frac{1}{6} b_1(t) \right) x^2 \]
\[ + \left( -\frac{1}{12} b_2(t) + \frac{1}{12} b_0(t) - \frac{1}{12} a_0(t) + \frac{1}{12} \right) x^4 + \ldots = 0. \]

It follows easily that:
\[ a_0(t) = 2, \quad a_1(t) = 0, \quad a_2(t) = 0, \quad a_3(t) = 0, \ldots \]
\[ b_0(t) = 1, \quad b_1(t) = 0, \quad b_2(t) = 0, \quad b_3(t) = 0, \ldots \]

Therefore:
\[ u(x, t) = x^2 + t^2, \]
\[ v(x, t) = \frac{x^2 + t^2}{2}. \]

Solving of this equation by HPM leads to an approximation solution, whilst we obtain an exact solution by applying the NHPM.

3. Conclusions

In this work, the new homotopy perturbation method has been proposed and compared with the homotopy perturbation method for solving two-dimensional nonlinear wave equations. We have used the first approximate solution to reach the exact solution of the problem. The Computations lead to a set of nonlinear equations. This set can be readily solved using Maple and putting these values into the first approximate solution yields the exact solution. This method has been applied to three examples successfully, and exact solutions of the equations are achieved, where traditional HPM leads to an approximate solution. We hope that this approach to two-dimensional nonlinear wave equations will be helpful in solving other nonlinear equations.

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References


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