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Dynamic Oligopolies with Production Adjustment Costs

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Single-product oligopolies, without product differentiation, are examined under the assumption that any increase in production levels has additional cost to the firms. Therefore, the best response of each firm depends on the current output of the rest of the industry and on the previous output of the firm. Two dynamic models are introduced. In the first case, the firms form adaptive expectations on the output of the rest of the industry and select the best response output levels and, in the second case, it is assumed that they adjust their output levels adaptively. Conditions are derived in both cases for the asymptotic stability of the equilibrium.

INTRODUCTION

The classical Cournot model [1] and its variants and extensions have been the focus of researchers for a very long time. In most earlier studies, oligopolies were considered as static N-person games and the existence and uniqueness of the equilibrium were the main concern. Several model variants and extensions were examined, including single-product models, with and without product differentiation, rent-seeking, labor-managed and multi-product oligopolies, to mention but a few. In the dynamic extensions, both discrete and continuous time scales were considered and the asymptotic properties of the equilibrium were investigated. The most important earlier results in single-product oligopolies are summarized in [2] and their multi-product extensions, with several applications, are presented in [3].

With the development of the theory of oligopolies, the examined models became more complex and sophisticated. However, most of them were based on the assumptions that all products are sold at each time period; the cost functions and the capacity of the firms, as well as the market demand function, remain the same during the entire time horizon under consideration and the market demand of the consecutive time periods are independent of each other. Based on realizing these shortcomings, an extensive research has recently been performed in order to develop and examine more realistic models. The book by Bischi, Chiarella, Kopel and Szidarovszky [4] presents the most recent developments in this field. Some simple earlier results are reported in Okuguchi and Szidarovszky [3].

In this paper, one of the important issues will be addressed, when output adjustment costs will be taken into consideration. Any increase in production levels usually requires an increase in capacity limits, which cannot be done without purchasing or renting new machinery and equipment, hiring more labor, or even building a new plant. All of these are costly and these costs have to be taken into account. First, dynamic oligopoly models will be developed, including production adjustment costs and, then, the asymptotic behavior of the resulting dynamic systems will be examined.

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MATHEMATICAL MODEL AND BEST RESPONSES

An N-firm single product oligopoly, without product differentiation, is considered. Let f denote the price function (or inverse demand function), C_k the production cost function of firm k and x_k the output of this firm. Assume discrete time scales and consider time period $t+1$. Then, the profit of firm k can be given as the difference between its revenue and costs. Let $Q_k = \sum_{l \neq k} x_l$ denote the output of the rest of the industry, then:

$$\Pi_k = x_k f(x_k + Q_k(t+1)) \quad C_k(x_k) \quad K_k(x_k \quad x_k(t)), \tag{1}$$

where $K_k(x_k \quad x_k(t))$ is the output adjustment cost. If $x_k > x_k(t)$, then, this is the cost of the output increase. If $x_k < x_k(t)$, then, there is some cost attached to idle machinery and layed-off labor etc. In this study, however, only the cost of output increase will be considered, so $K_k(\Delta) = 0$ as $\Delta \leqq 0$.

Assume that functions f, C_k and K_k are twice continuously differentiable in $\mathbb{R}_+$ and $K'_k(0) = K''_k(0)$, furthermore;

(A) $f' < 0, C'_k \geqq 0, K'_k \geqq 0$,

(B) $f' + x_k f'' < 0$,

(C) $f' \quad C''_k \quad K''_k < 0$,

for all k and feasible values of the involved variables.

At time period $t+1$, when firm k maximizes its profit, it does not know the simultaneous output decisions of the competitors, therefore, it forms an expectation, $Q_k^E(t+1)$, based on earlier observations and maximizes its expected profit as follows:

$$x_k f(x_k + Q_k^E(t+1)) \quad C_k(x) \quad K_k(x_k \quad x_k(t)). \tag{2}$$

With fixed values of $Q_k^E(t+1)$ and $x_k(t)$, this function is strictly concave in x_k. Assume, in addition, that each firm, k, has a finite maximal possible output capacity, L_k, then, the feasible strategy set of firm k is the compact set, $[0, L_k]$, and there is a unique response function, $\mathcal{R}_k(Q_k, x_k)$, which can be obtained as follows:

1. If $f(Q_k) \quad C'_k(0) \leqq 0$, then $\mathcal{R}_k(Q_k, x_k) = 0$,
2. If $L_k f'(L_k + Q_k) + f(L_k + Q_k) \quad C'_k(L_k) \quad K'_k(L_k \quad x_k) \geqq 0$, then $R_k(Q_k, x_k) = L_k$,
3. Otherwise, $z_k = \mathcal{R}_k(Q_k, x_k)$ is the unique solution of the monotonic equation:

$$z_k f'(z_k + Q_k) + f(z_k + Q_k) \quad C'_k(z_k) \quad K'_k(z_k \quad x_k) = 0, \tag{3}$$

inside interval $(0, L_k)$.

Assume that $\mathcal{R}_k(Q_k, x_k)$ is interior, that is, the third case applies. By implicitly differentiating Equation 3, with respect to Q_k and x_k, one has:

$$\mathcal{R}'_{kQ}.f' + z_k.f''.(1 + \mathcal{R}'_{kQ}) + f'.(1 + \mathcal{R}'_{kQ}) \quad C''_k.\mathcal{R}'_{kQ} \quad K''_k.\mathcal{R}'_{kQ} = 0,$$

and:

$$\mathcal{R}'_{kx}.f' + z_k.f''.\mathcal{R}'_{kx} + f'.\mathcal{R}'_{kx} \quad C''_k.\mathcal{R}'_{kx} \quad K''_k.(\mathcal{R}'_{kx} \quad 1) = 0,$$

which imply that:

$$\mathcal{R}'_{kQ} = \quad \frac{f' + z_k f''}{2f' + z_k f'' \quad C''_k \quad K''_k}, \tag{4}$$

and:

$$\mathcal{R}'_{kx} = \quad \frac{K''_k}{2f' + z_k f'' \quad C''_k \quad K''_k}. \tag{5}$$

Assumptions (A)-(C) imply that:

$$1 < \mathcal{R}'_{kQ} < 0 \leq \mathcal{R}'_{kx} < 1, \tag{6}$$

and:

$$1 < \mathcal{R}'_{kQ} \quad \mathcal{R}'_{kx}. \tag{7}$$

By assuming that the firms use adaptive learning about the output of the rest of the industry, one has the following $2N$-dimensional nonlinear dynamic system:

$$x_k(t+1) = \mathcal{R}_k\left(Q_k^E(t) + a_k.\left(\sum_{l \neq k} x_l(t) \quad Q_k^E(t)\right), x_k(t)\right), \tag{8}$$

$$Q_k^E(t+1) = Q_k^E(t) + a_k.\left(\sum_{l \neq k} x_l(t) \quad Q_k^E(t)\right), \tag{9}$$

where $a_k \in (0,1]$ is the speed of adjustment of firm k. Notice that $(\bar{x}_1, \cdots, \bar{x}_N, \bar{Q}_1^E, \cdots, \bar{Q}_N^E)$ is a steady of this system, if and only if, $(\bar{x}_1, \cdots, \bar{x}_N)$ is a Nash equilibrium of the N-firm oligopoly without output adjustment costs and for all k, $\bar{Q}_k^E = \sum_{l \neq k} \bar{x}_l$.

An alternative dynamic model can be developed by assuming that the firms use static expectations and adjust their outputs adaptively:

$$x_k(t+1) = x_k(t) + a_k\left(\mathcal{R}_k\left(\sum_{l \neq k} x_l(t), x_k(t)\right) \quad x_k(t)\right), \tag{10}$$

where $0 < a_k \leq 1$ is the speed of adjustment. This adjustment scheme means that the firms adjust their output only in the direction towards their best choices, instead of choosing best responses. This is a realistic approach in cases when larger output changes need more time than one period, or, when the firms want to react to market changes only gradually.

ASYMPTOTIC ANALYSIS

The asymptotic stability of Systems 8, 9 and 10 is examined by linearization. The Jacobian of Systems 8 and 9 can be written as follows:

$$\begin{pmatrix} r_{1x} & r_{1Q}a_1 & \cdots & r_{1Q}a_1 & r_{1Q}(1 \ \ a_1) & & & \\ r_{2Q}a_2 & r_{2x} & \cdots & r_{2Q}a_2 & & r_{2Q}(1 \ \ a_2) & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ r_{NQ}a_N & r_{NQ}a_N & \cdots & r_{Nx} & & & & R_{NQ}(1 \ \ a_N) \\ 0 & a_1 & \cdots & a_1 & 1 \ \ a_1 & & & \\ a_2 & 0 & \cdots & a_2 & & 1 \ \ a_2 & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ a_N & a_N & \cdots & 0 & & & & 1 \ \ a_N \end{pmatrix}$$

where $r_{kQ} = \frac{\partial R_k}{\partial Q_k}$ and $r_{kx} = \frac{\partial R_k}{\partial x_k}$ at the equilibrium.

The eigenvalue equation of this matrix can be written as follows:

$$r_{kx}u_k + r_{kQ}a_k \sum_{l \neq k} u_l + r_{kQ}(1 \quad a_k)v_k = \lambda u_k, \qquad (11)$$

$$a_k \sum_{l \neq k} u_l + (1 \quad a_k)v_k = \lambda v_k \quad (1 \leqq k \leqq n). \qquad (12)$$

Multiply Equation 12 by r_{kQ} and subtract the resulting equation from Equation 11 in order to have:

$$r_{kx}u_k = \lambda(u_k \quad r_{kQ}v_k),$$

which implies that:

$$r_{kQ}v_k = \frac{\lambda \quad r_{kx}}{\lambda}u_k,$$

where it is assumed that $\lambda \neq 0$, since a zero eigenvalue cannot destroy stability. By substituting this relation into Equation 11, one gets the following equation:

$$r_{kQ}a_k \sum_{l \neq k} u_l + \left(r_{kx} + \frac{(1 \quad a_k)(\lambda \quad r_{kx})}{\lambda} \quad \lambda \right) u_k = 0, \qquad (13)$$

for all k. This system has a nontrivial solution if and only if, its determinant is zero. That is, with the notation:

$$A_k(\lambda) = r_{kx} + \frac{(1 \quad a_k)(\lambda \quad r_{kx})}{\lambda} \quad \lambda,$$

and:

$$B_k(\lambda) = r_{kQ}a_k,$$

one has:

$$0 = \det \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & \cdots & B_1(\lambda) \\ B_2(\lambda) & A_2(\lambda) & \cdots & B_2(\lambda) \\ \vdots & \vdots & & \vdots \\ B_N(\lambda) & B_N(\lambda) & \cdots & A_N(\lambda) \end{pmatrix}$$

$$= \prod_{k=1}^{N} (A_k(\lambda) \quad B_k(\lambda)) . \left[1 + \sum_{k=1}^{N} \frac{B_k(\lambda)}{A_k(\lambda) \quad B_k(\lambda)} \right]$$

$$= 0,$$

where the determinantal identity used earlier in [5] is used.

First, the roots of the following equation are examined:

$$A_k(\lambda) \quad B_k(\lambda) = 0, \qquad (14)$$

which is the quadratic equation:

$$\lambda^2 + \lambda\left(a_k(1 + r_{kQ}) \quad r_{kx} \quad 1\right) + r_{kx}(1 \quad a_k) = 0. \qquad (15)$$

Then, it is proven that this equation has two real roots. Its discriminant is as follows:

$$(a_k z \quad r_{kx} \quad 1)^2 \quad 4r_{kx}(1 \quad a_k),$$

with $z = 1 + r_{kQ} \in (0, 1)$, which is a quadratic polynomial in z:

$$z^2 a_k^2 \quad 2za_k(r_{kx} + 1) + ((r_{kx} \quad 1)^2 + 4a_k r_{kx}).$$

This is a convex parabola with derivative:

$$2za_k^2 \quad 2a_k(r_{kx} + 1) < 2a_k^2 \quad 2a_k = 2a_k(a_k \quad 1) \leq 0,$$

since $z < 1, a_k \in (0, 1]$ and $0 \leq r_{kx} < 1$. Therefore, it is decreasing in the interval $(0, 1)$. Its value at $z = 1$ equals:

$$a_k^2 \quad 2a_k(r_{kx} + 1) + (r_{kx} \quad 1)^2 + 4a_k r_{kx}$$

$$= a_k^2 + 2a_k(r_{kx} \quad 1) + (r_{kx} \quad 1)^2$$

$$= (a_k + r_{kx} \quad 1)^2 \geq 0.$$

Therefore, the determinant is positive, so there are two real roots. Next, it is shown that these roots are between 1 and +1. Clearly,

$$r_{kx}(1 \quad a_k) < 1,$$

and, furthermore:

$$a_k(1 + r_{kQ}) \quad r_{kx} \quad 1 + r_{kx}(1 \quad a_k) + 1 > 0,$$

since the left hand side simplifies as follows:

$$a_k(1 + r_{kQ} \quad r_{kx}),$$

which is positive from Equation 7. In addition:

$$a_k(1 + r_{kQ}) + r_{kx} + 1 + r_{kx}(1 \quad a_k) + 1 > 0,$$

since the left hand side equals:

$$[1 \quad a_k(1 + r_{kQ})] + [1 + r_{kx}(2 \quad a_k)],$$

with both bracketed expressions being positive. Hence, all roots of Equation 14 are real and between 1 and +1.

Consider the following equation:

$$1+\sum_{k=1}^{N}\frac{B_k(\lambda)}{A_k(\lambda)-B_k(\lambda)}=0. \qquad (16)$$

The equilibrium is locally asymptotically stable, if all roots of this equation are inside the unit circle. In contrast to Equation 14, there is no guarantee that the roots are real and inside the unit circle. In general cases, computer methods can be used, however, in the case of symmetric firms, one can obtain simple analytic results. So, assuming that $a_k \equiv a, r_{kQ} \equiv r$ and $r_{kx} \equiv \bar{r}$, then, Equation 16 can be rewritten as follows:

$$A(\lambda)+(N-1)B(\lambda)=0,$$

that is,

$$\lambda^2+\lambda(-\bar{r}+a-1-ra(N-1))+\bar{r}(1-a)=0.$$

The roots are inside the unit circle if and only if:

$$\bar{r}(1-a)<1, \qquad (17)$$

$$(-\bar{r}+a-1-ra(N-1))+\bar{r}(1-a)+1>0, \qquad (18)$$

and:

$$(\bar{r}-a+1+ra(N-1))+\bar{r}(1-a)+1>0. \qquad (19)$$

Notice first that Equation 17 clearly holds, since $a \in (0,1]$ and $\bar{r} \in (0,1)$. Furthermore, Equation 18 can be simplified as follows:

$$a(1-\bar{r}-r(N-1))>0,$$

which also holds for all $a \in (0,1], \bar{r} \in [0,1)$ and $r \in (-1,0)$. Inequality 19 can be rewritten as follows:

$$a(1-r(N-1)+\bar{r})<2+2\bar{r}.$$

The multiplier of a is positive, so the roots are inside the unit circle if and only if:

$$a<\frac{2+2\bar{r}}{1+\bar{r}-r(N-1)}. \qquad (20)$$

Hence, the symmetric equilibrium is locally asymptotically stable, if the value of a is sufficiently small.

Returning to System 10, notice first that its Jacobian has a special structure as follows:

$$\begin{pmatrix} 1-a_1(1-r_{1x}) & a_1r_{1Q} & \cdots & a_1r_{1Q} \\ a_2r_{2Q} & 1-a_2(1-r_{2x}) & \cdots & a_2r_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ a_Nr_{NQ} & a_Nr_{NQ} & \cdots & 1-a_N(1-r_{Nx}) \end{pmatrix}.$$

It is easy to see (similarly to the previous case) that the characteristic polynomial of this matrix is as follows:

$$\prod_{k=1}^{N}(1-a_k(1-r_{kx})-\lambda-a_kr_{kQ}) \cdot \left[1+\sum_{k=1}^{N}\frac{a_kr_{kQ}}{1-a_k(1-r_{kx})-\lambda-a_kr_{kQ}}\right]=0. \qquad (21)$$

By using Equation 7 the root of the equation:

$$1-a_k(1-r_{kx})-\lambda-a_kr_{kQ}=0,$$

is:

$$\lambda=1-a_k(1-r_{kx}+r_{kQ})<1.$$

So, this root is inside the unit circle if:

$$a_k<\frac{2}{1-r_{kx}+r_{kQ}}. \qquad (22)$$

If $g(\lambda)$ denotes the bracketed term, then, the poles are the values $1-a_k(1-r_{kx})-a_kr_{kQ}$; the right hand side limit at each pole is $+\infty$ and the left hand side limit is $-\infty$. In addition, $g(\lambda)$ tends to 1, as $\lambda \to +\infty$ or $\lambda \to -\infty$. Furthermore, $g'(\lambda)<0$. Therefore, there is one root before the smallest pole and one between each pair of consecutive poles. Therefore, all roots are real and they are between -1 and $+1$, if Equation 22 holds for all k and:

$$\sum_{k=1}^{N}\frac{a_kr_{kQ}}{2-a_k(1+r_{kQ}-r_{kx})}>-1, \qquad (23)$$

that is, all values, a_k, are sufficiently small.

Consider next symmetric firms, when $a_k \equiv a, r_{kQ} \equiv r$ and $r_{kx} \equiv \bar{r}$. In this case, Conditions 22 and 23 can be rewritten as follows:

$$a<\frac{2}{1-\bar{r}+r}, \qquad (24)$$

and:

$$a<\frac{2}{1-\bar{r}+r(1-N)}. \qquad (25)$$

Notice that Inequality 25 is stronger than Relation 24, so it is the stability condition.

CONCLUSION

In this paper, single-product oligopolies were examined without product differentiation and with production adjustment costs. Under realistic conditions, the unique best response of each firm, as the function of the current output of the rest of the industry and the previous output of the firm, was determined. Two dynamic models were developed. In the first case, each firm used adaptive expectations for the output of the rest of the industry and changed its output level to the corresponding best response level. In the second case, it is assumed that the firms used static expectations and adjusted their output levels adaptively.

Conditions were derived for the asymptotic stability of the steady state. These conditions are always satisfied, if the speeds of adjustments are sufficiently small.

REFERENCES

1. Cournot, A., *Recherches sur les Principes Mathématiques de la Théorie de Richessess*, Hachette, Paris (1938). (English translation: *Researches into the Mathematical Principles of the Theory of Wealth*, Kelley, New York (1960)).
2. Okuguchi, K., *Expectations and Stability in Oligopoly Models*, Springer-Verlag, Berlin/Heidelberg/New York (1976).
3. Okuguchi, K. and Szidarovszky, F., *The Theory of Oligopoly with Multi-Product Firms*, 2nd Edn., Springer-Verlag, Berlin/Heidelberg/New York (1999).
4. Bischi, G.I., Chiarella, C., Kopel, M. and Szidarovszky, F., *Nonlinear Oligopolies: Stability and Bifurcations* (2008).
5. Chiarella, C. and Szidarovszky, F. "The birth of limit cycles in nonlinear oligopolies with continuously distributed information lags", *Modelling Uncertainty*, M. Dror, P. L'Ecyer and F. Szidarovszky, Eds., Kluwer Academic Publishers, Dordrecht, pp 249-268 (2001).