EXISTENCE OF EXTREMAL SOLUTIONS FOR IMPULSIVE DELAY FUZZY INTEGRODIFFERENTIAL EQUATIONS IN n-DIMENSIONAL FUZZY VECTOR SPACE

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Abstract. In this paper, we study the existence of extremal solutions for impulsive delay fuzzy integrodifferential equations in n-dimensional fuzzy vector space, by using monotone method. We show that obtained result is an extension of the result of Rodríguez-López [8] to impulsive delay fuzzy integrodifferential equations in n-dimensional fuzzy vector space.

1. Introduction

Fuzzy theory has developed engineering, economics, agriculture, computers, etc. in various fields by many scholars since 1965. Moreover fuzzy integrodifferential equations are a field of increasing interest, due to their applicability to the analysis of phenomena where imprecision in inherent.


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In this paper, we study the existence of extremal solutions for the following impulsive delay fuzzy integrodifferential equations in fuzzy vector space.

\[
\begin{align*}
\frac{dx(t)}{dt} &= f_i(t, x(t), \int_0^t q_i(t, s, x(s))ds), \quad t \in J, \\
x_i(t) &= \phi_i(t), \quad t \in [-r, 0], \\
x_i(t_k^+) &= I_k(x_i(t_k)), \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n, 
\end{align*}
\]  

where \( T > 0, \ J = [0, T], \ 0 < t_1 < \ldots < t_m < t_{m+1} = T, \) \( E_N^i \) is the set of all upper semi-continuously convex fuzzy numbers on \( R \) with \( E_N^i \neq E_N^j, \) \( i \neq j, \)

\( f_i : J \times E_N^i \times E_N^j \rightarrow E_N^k \) and \( q_i : J \times E_N^i \rightarrow E_N^j \) are regular continuous fuzzy function, \( (x_i)_0 = x_i(t + \theta), \ \theta \in [-r, 0]. \phi_i \in C([-r, 0], E_N^i) \) is initial function and \( I_k \in C(E_N^i, E_N^j) \) are bounded functions.

2. Preliminaries

In this section, we give basic definitions, terminologies, notations and Lemmas which are most relevant to our investigated and are needed in later chapters. All undefined concepts and notions used here are standard.

A fuzzy set of \( R^n \) is a function \( u : R^n \rightarrow [0, 1]. \) For each fuzzy set \( u, \) we denote by \( [u]^\alpha = \{ x \in R^n : u(x) \geq \alpha \} \) for any \( \alpha \in [0, 1], \) its \( \alpha \)-level set and \( [u]^0 = cl\{ x \in R^n : u(x) > 0 \} \) (the closure of \( \{ x \in R^n : u(x) > 0 \} \)). Let \( u, v \) be fuzzy sets of \( R^n. \) It is well known that \( [u]^\alpha = [v]^\alpha \) for each \( \alpha \in [0, 1] \) implies \( u = v. \) Let \( E^n \) denote the collection of all fuzzy sets of \( R^n \) that satisfies the following conditions:

1. \( u \) is normal, i.e., there exists an \( x_0 \in R^n \) such that \( u(x_0) = 1; \)
2. \( u \) is fuzzy convex, i.e., \( u(\lambda x + (1 - \lambda) y) \geq \min\{ u(x), u(y) \} \) for any \( x, y \in R^n, \ 0 \leq \lambda \leq 1; \)
3. \( u(x) \) is upper semi-continuous, i.e., \( u(x_0) = \lim_{k \rightarrow \infty} u(x_k) \) for any \( x_k \in R^n, \)
4. \( [u]^0 \) is compact.

We call \( u \in E^n \) a \( n \)-dimension fuzzy number.

Wang et al. [11] defined \( n \)-dimension fuzzy vector space and investigated its properties.

For any \( u_i \in E, \ i = 1, 2, \ldots, n, \) we call the ordered one-dimension fuzzy number class \( u_1, u_2, \ldots, u_n \) (i.e., the Cartesian product of one-dimension fuzzy number \( u_1, u_2, \ldots, u_n \) a \( n \)-dimension fuzzy vector, denote it as \( (u_1, u_2, \ldots, u_n), \) and call the collection of all \( n \)-dimension fuzzy vectors (i.e., the Cartesian product \( E \times E \times \cdots \times E \)) \( n \)-dimension fuzzy vector space, and denote it as \((E)^n).\)

**Definition 2.1** [12] If \( u \in E^n, \) and \( [u]^\alpha \) is a hyperrectangle, i.e., \([u]^\alpha \) can be represented by \( \prod_{i=1}^n [u_i^\alpha, u_i^\alpha], \) i.e., \( [u_1^\alpha, u_1^\alpha] \times [u_2^\alpha, u_2^\alpha] \times \cdots \times [u_n^\alpha, u_n^\alpha] \) for every \( \alpha \in [0, 1], \) when \( u_1^\alpha \leq u_1^\alpha, \) then we call \( u \) a fuzzy \( n \)-cell number. We denote the collection of all fuzzy \( n \)-cell numbers by \( L(E^n). \)
Theorem 2.2. [11] For any \( u \in L(E^n) \) with \( [u]^\alpha = \prod_{i=1}^n [u_{i\alpha}, u_{i\alpha}] \) (\( \alpha \in [0, 1] \)), there exists a unique \( (u_1, u_2, \ldots, u_n) \in (E)^n \) such that \( [u_i]^\alpha = [u_{i\alpha}, u_{i\alpha}] \) (\( i = 1, 2, \ldots, n \) and \( \alpha \in [0, 1] \)). Conversely, for any \( (u_1, u_2, \ldots, u_n) \in (E)^n \) with \( [u_i]^\alpha = [u_{i\alpha}, u_{i\alpha}] \) (\( i = 1, 2, \ldots, n \) and \( \alpha \in [0, 1] \)), there exists a unique \( u \in L(E^n) \) such that \( [u]^\alpha = \prod_{i=1}^n [u_{i\alpha}, u_{i\alpha}] \) (\( \alpha \in [0, 1] \)).

Remark 2.3. [11] Theorem 2.2 indicates that fuzzy \( n \)-cell numbers and \( n \)-dimension fuzzy vectors can represent each other, so \( L(E^n) \) and \( (E)^n \) may be regarded as identity. If \( (u_1, u_2, \ldots, u_n) \in (E)^n \) is the unique \( n \)-dimension fuzzy vector determined by \( u \in L(E^n) \), then we denote \( u = (u_1, u_2, \ldots, u_n) \).

Let \( (E_N)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n \), \( E_N^i (i = 1, 2, \ldots, n) \) is fuzzy subset of \( R \). Then \( (E_N)^n \subseteq (E)^n \).

Definition 2.4. [12] The complete metric \( D_L \) on \( (E_N^i)^n (i = 1, 2, \ldots, n) \) is defined by

\[
D_L(u, v) = \sup_{0 < \alpha \leq 1} \frac{d_L([u]^\alpha, [v]^\alpha)}{\alpha}
\]

for any \( u, v \in (E_N^i)^n \), which satisfies \( d_L(u + w, v + w) = d_L(u, v) \).

Definition 2.5. Let \( u, v \in C([-r, T], (E_N^i)^n) \)

\[
H_1(u, v) = \sup_{-r < t \leq T} D_L(u(t), v(t)).
\]

Definition 2.6. The derivative \( x'(t) \) of a fuzzy process \( x \in (E_N^i)^n \) is defined by

\[
[x'(t)]^\alpha = \prod_{i=1}^n [(x_{i\alpha}'(t))', (x_{i\alpha}'(t))']
\]

provided that equation defines a fuzzy \( x'(t) \in (E_N^i)^n \).

Definition 2.7. The fuzzy integral \( \int_a^b x(t)dt, a, b \in [0, T] \) is defined by

\[
\left[ \int_a^b x(t)dt \right]^\alpha = \prod_{i=1}^n \left[ \int_a^b x_{i\alpha}^i(t)dt \right], \int_a^b x_{i\alpha}^i(t)dt
\]

provided that the Lebesgue integrals on the right hand side exist.

Definition 2.8. [7] Let \( x, y \in E^1 \). We say that \( x \leq y \) if and only if \( x_{i\alpha} \leq y_{i\alpha} \) for every \( \alpha \in [0, 1] \).

Definition 2.9. Let \( x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in (E_N^i)^n \). We say that \( x \leq y \) if for \( x_i, y_i \in E_N^i, i = 1, 2, \ldots, n \),

\[
x_i \leq y_i.
\]
Lemma 2.10. [8] If \( \{ f_n \} \subseteq C([c, d], E^1) \), \( g \in C([c, d], E^1) \) are such that 
\[ f_n \leq g, \ \forall n \in \mathbb{N} , \]
and \( f_n(t) \) converges to \( f(t) \) in \( E^1 \), for all \( t \in [c, d] \), then \( f \leq g \).

Definition 2.11. [12] Let \( x, y \in E^n \). If there exists \( z \in E^n \) such that \( x = y + z \) then we call \( x, y \) having Hukuhara difference and \( y \) is called the Hukuhara difference of \( x \) and \( y \), denoted \( x - y \).

Definition 2.12. [2] A fuzzy set \( u \in E^n \) is called a Lipschitz fuzzy set if it is a Lipschitz function of its membership grade in the sense that
\[ d_H([u]^\alpha, [u]^\beta) \leq K|\alpha - \beta| \]
for all \( \alpha, \beta \in [0, 1] \) and some fixed, finite constant \( K \).

Lemma 2.13. [8] Let \( I \) be a closed interval in \( R \), and \( B \subseteq C(I, E^1) \) such that, for all \( x \in B \) and \( t \in I \), \( x(t) \) is a continuous fuzzy number. Consider
\[
\overline{B}_L = \{ \pi_L : x \in B \} \subseteq C([0, 1] \times I, R), \\
\overline{B}_R = \{ \pi_R : x \in B \} \subseteq C([0, 1] \times I, R),
\]
where
\[
\pi_L : [0, 1] \times I \to R, \\
(a, t) \to \pi_L(a, t) = (x(t))_L(a) = (x(t))_{al},
\]
and
\[
\pi_R : [0, 1] \times I \to R, \\
(a, t) \to \pi_R(a, t) = (x(t))_R(a) = (x(t))_{ar}.
\]
if \( \overline{B}_L \) and \( \overline{B}_R \) are relatively compact sets in \( (C([0, 1] \times I, R), \| \cdot \|_\infty) \), then \( B \) is a relatively compact set in \( C(I, E^1) \).

3. Existence of Extremal Solutions

In order to prove the existence of extremal solutions for equations in \( n \)-dimensional fuzzy vector space, we define
\[
f = (f_1, f_2, \ldots, f_n), \\
q = (q_1, q_2, \ldots, q_n), \\
x_i = (x_1(t), x_2(t), \ldots, x_n(t)), \\
x(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \\
\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t)),
\]
then \( f, q, x_i, x(t), \phi(t) \in (E^1_N)^n, \ i = 1, 2, \ldots, n \).

We consider the following impulsive delay fuzzy integro-differential equations in fuzzy vector space \( (E^1_N)^n \):
\[
\begin{cases}
\frac{dx(t)}{dt} = f(t, x(t), \int_0^t q(t, s, (x)_s)ds), \ t \in J, \\
x(t) = \phi(t), \ t \in [-r, 0], \\
x(t^+_k) = I_k(x(t_k)), \ t \neq t_k, \ k = 1, 2, \ldots, m,
\end{cases}
\]
where \( T > 0, J = [0, T], 0 < t_1 < \cdots < t_m < t_{m+1} = T, \) \( f : J \times (E_N^i)^n \times (E_N^i)^n \to (E_N^i)^n \) and \( q : J \times J \times (E_N^i)^n \to (E_N^i)^n \) are regular continuous fuzzy function, \( x_t = x(t + \theta), \theta \in [-r, 0], \phi \in C([-r, 0], (E_N^i)^n) \) is initial function and \( I_k \in C((E_N^i)^n, (E_N^i)^n) \) are bounded functions.

To define solutions for the impulsive fuzzy integrodifferential equations, we consider the following space:

\[
\Omega_i = \left\{ x_t : J \to E_N^i : (x_t)_k \in C(J_k, E_N^i), J_k = (t_k, t_{k+1}], \text{and there exist } x_t(0^+), x_t(T^-), x_t(t_k^+) = x_t(t_k), (k = 1, 2, \ldots, m, \ i = 1, 2, \ldots, n) \right\}.
\]

Let \( \Omega_i' = \Omega_i \cap C([-r, T], E_N^i), \ i = 1, 2, \ldots, n, \) and \( \Omega' = \prod_{i=1}^n \Omega_i' \).

**Definition 3.1.** For the partial ordering \( \leq_n \), a function \( a \in \Omega' \) is a \( \leq_n \)-lower solution for (2) if

\[
\begin{align*}
& a_t' \leq_n f(t, a_t, \int_0^t q(t, s, a_s)ds), t \in J, \\
& a(t) \leq_n \phi(t), t \in [-r, 0], \\
& a(t_k^+) \leq_n I_k(a(t_k)), k = 1, 2, \ldots, m,
\end{align*}
\]

we define \( a \) as a \( \leq_n \)-upper solution for (2) as a function satisfying the reverse inequalities.

To find extremal solutions for equations (2) by using monotone method, for \( M > 0 \), let

\[
\frac{dx(t)}{dt} = Mx(t) + f\left(t, x(t), \int_0^t q(t, s, x_s)ds\right) - Mx(t).
\]

And if \( F\left(t, x_t, \int_0^t q(t, s, x_s)ds\right) = f\left(t, x_t, \int_0^t q(t, s, x_s)ds\right) - Mx(t), \) then we consider the following equations:

\[
\begin{align*}
& \frac{dx(t)}{dt} = Mx(t) + F\left(t, x(t), \int_0^t q(t, s, x_s)ds\right), t \in J, \\
& x(t) = \phi(t), t \in [-r, 0], \\
& x(t_k^+) = I_k(x(t_k)), t \neq t_k, k = 1, 2, \ldots, m.
\end{align*}
\]

We define \( \hat{x}_t(\theta) : [-r, T] \to (E_N^i)^n \) for \( x_t(\phi) \in C(J, (E_N^i)^n), \theta \in [-r, 0], \)

\[
\hat{x}_t(\phi) = \begin{cases} 
\phi(t), & t + \theta \in [-r, 0], \\
x_t(\phi), & t + \theta \in J.
\end{cases}
\]

Assume the following:

(FL) For \( \hat{x}, \hat{y}, \hat{\eta}, \hat{\zeta}, \hat{u}, \hat{v} \in C([-r, T], (E_N^i)^n), \) there exist positive numbers \( h, k, p \) so that

\[
\begin{align*}
& d_L\left(\left[F\left(t, \hat{x}_t, \hat{u}_t\right)\right]^n, \left[F\left(t, \hat{y}_t, \hat{v}_t\right)\right]^n\right) \\
& \leq h d_L\left(\left[x_t^n, [y_t]^n\right], [u_t]^n, [v_t]^n\right), \\
& d_L\left(\left[q(t, s, \hat{\eta}_s)\right]^n, [q(t, s, \hat{\zeta}_s)]^n\right) \leq p d_L\left(\left[\eta_t^n, \xi_t^n\right], \left[\xi_t^n\right]^n\right),
\end{align*}
\]

and \( F(t, X_{[1]}(0), X_{[1]}(0)) = X_{[0]}(0) \).
(F2) For \( x, y \in C(J : (E^n)^\alpha) \), there exists a \( d > 0, k = 1, 2, \cdots, m \), so that
\[
d_L\left([I_k(x(t_k))]^\alpha, [I_k(y(t_k))]^\alpha\right) \leq d d_L\left([x(t)]^\alpha, [y(t)]^\alpha\right),
\]
and \( I_k(\mathcal{X}(0)) = \mathcal{X}(0) \).

(F3) \( c(d + hT + kpT^2) < 1 \).

**Lemma 3.2.** If \( x \in \Omega' \) is an integral solution of \( (3) \), then \( x \in \Omega' \) is given by
\[
x(t) = x(t)(\phi(0)) = S(t)\phi(0) + \int_0^t S(t-s)F(s, \hat{x}_s(\phi), \int_0^s q(s, \tau, \hat{x}_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t} S(t-t_k)I_k(x(t_k)), t \in J,
\]
where \( S(t) = \exp\left\{ \int_0^t Mdt \right\} \) is continuous with \( |S(t)| \leq c, c > 0 \), for all \( t \in J \).

**Theorem 3.3.** If hypotheses (F1)-(F3) are hold. Then the equations \( (3) \) have a unique solution \( x \in \Omega' \).

**Proof.** Let \( \delta > 0 \) satisfy
\[
d_L\left([S(t)\phi(0))]^\alpha, [\phi(0)]^\alpha\right) \leq \frac{\delta}{3},
\]
for \( t \in [−r, T] \) and
\[
c(d + hT + kpT^2) \sup_{t \in [−r, 0]} D_L\left(x(t), \mathcal{X}(0)(0)\right) \leq \frac{\delta}{3}.
\]
Let us define \( K \), the nonempty closed bounded subset of \( \in \Omega' \), by
\[
K = \left\{ \eta \in \Omega' \mid \eta(0) = \phi(0), H_1(\eta, \phi) \leq \delta \right\}.
\]

Define a mapping \( G \) on \( K \) by
\[
Gx(t)(\phi) = S(t)\phi(0) + \int_0^t S(t-s)F(s, \hat{x}_s(\phi), \int_0^s q(s, \tau, \hat{x}_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t} S(t-t_k)I_k(x(t_k)).
\]
For \( G\hat{x}_t(\phi) \), if \( −r \leq t + \theta \leq 0 \), then \( G\hat{x}_t(\phi)(\theta) = \phi(t + \theta) \) and hence
\[
H_1(G\hat{x}_t, \phi) \leq \frac{\delta}{3}.
\]
If \( 0 < t + \theta \leq T \), then
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\[ d_L \left( [G\hat{x}_s(\phi)(\theta)]^\alpha, [\phi(\theta)]^\alpha \right) \]

\[ = d_L \left( [S(t + \theta)\phi(0) + \int_0^{t+\theta} S(t + \theta - s)F(s, \hat{x}_s(\phi), \int_0^s q(s, \tau, \hat{x}_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t + \theta} S(t + \theta - t_k)I_k(x(t_k)) ]^\alpha, [\phi(\theta)]^\alpha \right) \]

\[ \leq d_L \left( [S(t + \theta)\phi(0)]^\alpha, [\phi(0)]^\alpha \right) + d_L \left( [\phi(0)]^\alpha, [\phi(0)]^\alpha \right) + \int_0^{t+\theta} d_L \left( [S(t + \theta - s)F(s, \hat{x}_s(\phi), \int_0^s q(s, \tau, \hat{x}_\tau(\phi))d\tau)]^\alpha, [S(t)F(s, \hat{x}_s(\phi), \int_0^s q(s, \tau, \hat{x}_\tau(\phi))d\tau)]^\alpha \right)ds \]

\[ + \sum_{0 < t_k < t + \theta} d_L \left( [S(t + \theta - t_k)I_k(x(t_k))]^\alpha, [S(t - t_k)I_k(x(t_k))]^\alpha \right) \]

\[ \leq \frac{\delta}{3} + \frac{\delta}{3} + c \int_0^{t+\theta} \left( k d_L \left( [x_s(\phi)]^\alpha, [x(\phi)]^\alpha \right) + k p \int_0^s d_L \left( [x_\tau(\phi)]^\alpha, [x(\phi)]^\alpha \right)d\tau \right)ds \]

\[ + c d L \left( [x(t)]^\alpha, [x(\phi)]^\alpha \right). \]

\[ D_L \left( G\hat{x}_s(\phi)(\theta), \phi(\theta) \right) \]

\[ = \sup_{0 < \alpha \leq 1} d_L \left( [G\hat{x}_s(\phi)(\theta)]^\alpha, [\phi(\theta)]^\alpha \right) \]

\[ \leq \frac{\delta}{3} + \frac{\delta}{3} + c \int_0^{t+\theta} \left( k d_L \left( [x_s(\phi)]^\alpha, [x(\phi)]^\alpha \right) + k p \int_0^s d_L \left( [x_\tau(\phi)]^\alpha, [x(\phi)]^\alpha \right)d\tau \right)ds \]

\[ + c d \sup_{0 < \alpha \leq 1} d_L \left( [x(t)]^\alpha, [x(\phi)]^\alpha \right) \]

\[ \leq \frac{\delta}{3} + \frac{\delta}{3} + c h \int_0^{t+\theta} D_L \left( x_s(\phi), x(\phi) \right)ds \]

\[ + c k p \int_0^{t+\theta} D_L \left( x_\tau(\phi), x(\phi) \right)d\tau + c d D_L \left( x(t), x(\phi) \right). \]
Hence

\[
H_1(G\tilde{x}_t, \phi) = \sup_{\theta \in [-r,0]} DL\left(G\tilde{x}_t(\phi), \phi(\theta)\right)
\]

\[
\leq \frac{\delta}{3} + \frac{\delta}{3} + ch \sup_{\theta \in [-r,0]} \int_0^{t+\theta} DL\left(x_s(\phi), X(0)_t(0)\right) ds
\]

\[
+ c\kappa T^2 \sup_{\theta \in [-r,0]} DL\left(x_s(\phi), X(0)_t(0)\right)
\]

\[
+ c \sup_{\theta \in [-r,0]} DL\left(x(t), X(0)_t(0)\right)
\]

\[
\leq \frac{\delta}{3} + \frac{\delta}{3} + c(d + hT + k\kappa T^2) \sup_{\theta \in [-r,0]} DL\left(x_s(\phi), X(0)_t(0)\right)
\]

\[
\leq \delta.
\]

Therefore we have

\[
G\tilde{x}_t(\phi) \in K, \text{ i.e. } G^t K \rightarrow K.
\]

Furthermore, if \(x_t(\phi), y_t(\phi) \in K\), then we have

\[
d_L(\left[G\tilde{x}_t(\phi)\right]^{\alpha}, [G\tilde{y}_t(\phi)\right]^{\alpha})
\]

\[
= d_L\left(\left[S(t + \theta)\phi(0) + \int_0^{t+\theta} S(t + \theta - s)F\left(s, \tilde{x}_s(\phi), \int_0^{\theta} q(s, \tau, \tilde{x}_\tau(\phi))d\tau\right) ds
\right.
\]

\[
+ \sum_{0 < t_k < t+\theta} S(t + \theta - t_k)I_k(x(t_k))\right]^{\alpha},
\]

\[
\left[S(t + \theta)\phi(0) + \int_0^{t+\theta} S(t + \theta - s)F\left(s, \tilde{y}_s(\phi), \int_0^{\theta} q(s, \tau, \tilde{y}_\tau(\phi))d\tau\right) ds
\right.
\]

\[
+ \sum_{0 < t_k < t+\theta} S(t + \theta - t_k)I_k(y(t_k))\right]^{\alpha}\right)
\]

\[
\leq \int_0^{t+\theta} d_L\left(\left[S(t + \theta - s)F\left(s, \tilde{x}_s(\phi), \int_0^{\theta} q(s, \tau, \tilde{x}_\tau(\phi))d\tau\right) ds
\right.
\]

\[
\left[S(t + \theta - s)F\left(s, \tilde{y}_s(\phi), \int_0^{\theta} q(s, \tau, \tilde{y}_\tau(\phi))d\tau\right) ds\right]^{\alpha} ds
\]
\[
+ d_L \left( \sum_{0 < t_k < t + \theta} S(t + \theta - t_k) I_k \{ x(t_k) \} \right)^\alpha,
\]
\[
\leq c \int_0^{t + \theta} \left( h d_L \left( [x_s(\phi)]^\alpha, [y_s(\phi)]^\alpha \right) + k p \int_0^\theta d_L \left( [x_\tau(\phi)]^\alpha, [y_\tau(\phi)]^\alpha \right) d\tau \right) ds
+ c d d_L \left( [x(t)]^\alpha, [y(t)]^\alpha \right).
\]

\[
D_L \left( G \tilde{\xi}_t(\phi), G \tilde{\eta}_t(\phi) \right)
= \sup_{0 < \alpha \leq 1} d_L \left( [G \tilde{\xi}_t(\phi)]^\alpha, [G \tilde{\eta}_t(\phi)]^\alpha \right)
\leq c h \int_0^{t + \theta} \sup_{0 < \alpha \leq 1} d_L \left( [x_s(\phi)]^\alpha, [y_s(\phi)]^\alpha \right) ds
+ c k p \int_0^{t + \theta} \sup_{0 < \alpha \leq 1} d_L \left( [x_\tau(\phi)]^\alpha, [y_\tau(\phi)]^\alpha \right) d\tau ds
+ c d \sup_{0 < \alpha \leq 1} d_L \left( [x(t)]^\alpha, [y(t)]^\alpha \right)
\leq c h \int_0^{t + \theta} D_L \left( x_s(\phi), y_s(\phi) \right) ds
+ c k p \int_0^{t + \theta} \int_0^\theta D_L \left( x_\tau(\phi), y_\tau(\phi) \right) d\tau ds
+ c d D_L \left( x(t), y(t) \right).
\]

Hence
\[
H_1 \left( G \tilde{\xi}_t, G \tilde{\eta}_t \right)
= \sup_{\theta \in [-r, 0]} D_L \left( G \tilde{\xi}_t(\phi), G \tilde{\eta}_t(\phi) \right)
\leq c h \sup_{\theta \in [-r, 0]} \int_0^{t + \theta} D_L \left( x_s(\phi), y_s(\phi) \right) ds
+ c k p \sup_{\theta \in [-r, 0]} \int_0^{t + \theta} \int_0^\theta D_L \left( x_\tau(\phi), y_\tau(\phi) \right) d\tau ds
+ c d \sup_{\theta \in [-r, 0]} D_L \left( x(t), y(t) \right)
\leq c h T \sup_{\theta \in [-r, 0]} D_L \left( x_t(\phi), y_t(\phi) \right)
+ c k p T^2 \sup_{\theta \in [-r, 0]} D_L \left( x_t(\phi), y_t(\phi) \right)
+ c d \sup_{\theta \in [-r, 0]} D_L \left( x(t), y(t) \right)
\leq c (d + h T + k p T^2) H_1 (x_t, y_t).
\]
By hypothesis (F3), $G$ is a contraction mapping. Using the Banach fixed point theorem, the equations (3) have a unique fixed point $x_t(\omega) \in K$. This complete the proof of theorem. \hfill \Box

Now, for showing the existence of extremal solutions of equations (2), the following Lemmas 3.4 and 3.5 are, respectively, extensions of Lemmas 2.10 and 2.13 (see [8]) to $n$-dimensional fuzzy vector space.

**Lemma 3.4.** Let $\{P_n\} = \{(P_n)_1 \times (P_n)_2 \times \cdots \times (P_n)_n\} \subset C([c, d], (E^n_N))$, $P = (P_1 \times P_2 \times \cdots \times P_n)$, $Q = (Q_1 \times Q_2 \times \cdots \times Q_n) \subset C([c, d], (E^n_N))$ are such that $P_n \leq Q_n$ and $P_n(t)$ converges to $P(t)$ in $(E^n_N)^n$, for all $t \in [c, d]$, then $P \leq Q$.

**Proof.** By Lemma 2.10, for each $i = 1, 2, \cdots, n$, $(P_n)_i \leq Q_i$ and $(P_n)_i(t)$ converges to $P_i(t)$ in $E^1$, then $P_i \leq Q_i$. Therefore $P_n \leq Q_n$ and $P_n(t)$ converges to $P(t)$ in $(E^n_N)^n$, for all $t \in [c, d]$, then $P \leq Q$. \hfill \Box

**Lemma 3.5.** Let $H = H_1 \times H_2 \times \cdots \times H_n \subset \Omega'$. We consider

$$
\overline{H}_i = \{x_i : x \in H\} \subseteq C([0, 1] \times [-r, T], R^n),
$$

$$
\overline{H}_r = \{x_r : x \in H\} \subseteq C([0, 1] \times [-r, T], R^n),
$$

where $x_i(\alpha, t) = (x(t))^\alpha_i$, $x_r(\alpha, t) = (x(t))^\alpha_r$. If $\overline{H}_i$ and $\overline{H}_r$ are relatively compact sets in $C([0, 1] \times [-r, T], R^n)$, then $H$ is a relatively compact set in $C([-r, T], (E^n_N)^n)$.

**Proof.** By Lemma 2.13, for each $i = 1, 2, \cdots, n$, $\overline{H}_{1i}$ and $\overline{H}_{ri}$ are relatively compact sets in $C([0, 1] \times [-r, T], R^i)$. Then

$$
\overline{H}_i = \overline{H}_{1i} \times \overline{H}_{2i} \times \cdots \times \overline{H}_{ni}
$$

and

$$
\overline{H}_r = \overline{H}_{1r} \times \overline{H}_{2r} \times \cdots \times \overline{H}_{nr}
$$

are relatively compact sets. In consequence, $H$ is a relatively compact set. \hfill \Box
Consider a function $H$ which satisfies:

\( d_L \left( [q(s, \tau, \hat{u}_t)]^a, [q(s, \tau, \hat{u}_t)]^b \right) \leq \frac{\varepsilon}{6cT} \) \tag{F7} 

For a fixed $\eta$, $\zeta \in A$ so we obtain that $\alpha - \beta \leq \delta_4 > 0$ such that $|\alpha - \beta| < \delta_4$, then $a, b \in \mathbb{R}$ and $a, b \in (A^\eta)_{\alpha, \beta}$.

We can now employ Lemma 3.5 with

$$H = \{(a_n)_t \in \Omega' : a_t \leq (a_n)_t \leq b_t\}.$$ 

Then the set $H$ is closed bounded convex.

**Lemma 3.6.** Consider a function $A$ defined by

$$A : [a, b]_n \rightarrow \Omega'$$

$$\eta \mapsto A\eta = x_\eta,$$

which satisfies:

(i) $A([a, b]_n) \subseteq [a, b]_n$,

(ii) $A$ is $\leq_n$-nondecreasing.

**Proof.** For a fixed $\eta \in [a, b]_n$, we consider the problem

\[
\begin{align*}
\dot{x}(t) &= M\eta(t) + F\left(t, \eta(t), \int_0^t q(t, s, \eta(s))ds\right), \quad t \in [0, T], \\
x(t) &= \phi(t), \quad t \in [-r, 0], \\
x(t^+_k) &= I_k(\eta(t_k)), \quad t \neq t_k, \quad k = 1, 2, \ldots, m,
\end{align*}
\]  

which has, by Theorem 3.3, a unique solution $x_\eta \in \Omega'$.

We prove that $A$ is $\leq_n$-nondecreasing. $Aa \geq_n a$, and $Ab \leq_n b$. Indeed, let $\eta, \zeta \in [a, b]$ be such that $\eta \leq_n \zeta$, then $A\eta$ and $A\zeta$ are functions in $\Omega'$ and

$$A(\eta)'(t) = M\eta(t) + F\left(t, \eta(t), \int_0^t q(t, s, \eta(s))ds\right)$$

$$\leq_n M\zeta(t) + F\left(t, \zeta(t), \int_0^t q(t, s, \zeta(s))ds\right) = (A\zeta)'(t), \quad t \in [0, T],$$

$$A(\eta)(t) = (A\phi)(t) \leq_n (A\phi)(t) = (A\zeta)(t), \quad t \in [-r, 0],$$

$$A(\eta)(t^+_k) = I_k(\eta(t_k)) \leq_n I_k(\zeta(t_k)) = (A\zeta)(t^+_k), \quad k = 1, 2, \ldots, m,$$

so we obtain that $A\eta \leq_n A\zeta$ on $[-r, T]$. Moreover, let $Aa \in \Omega'$, which satisfies, the properties of the $\leq_n$-lower solution and the partial ordering,

$$a'(t) \leq_n Ma(t) + F\left(t, a(t), \int_0^t q(t, s, a(s))ds\right) = (Aa)(t), \quad t \in [0, T],$$

$$a(t) \leq_n (A\phi)(t) = (Aa)(t), \quad t \in [-r, 0],$$

$$a(t^+_k) \leq_n I_k(a(t_k)) = (Aa)(t^+_k), \quad k = 1, 2, \ldots, m,$$

then $a \leq_n Aa$ on $[-r, T]$. Similarly, $b \geq_n Ab$ on $[-r, T]$. 

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This prove that \( A : [a, b] \to [a, b] \) and \( A \) is nondecreasing. Define the sequences \( \{a_n\}, \{b_n\} \) such that \( a_0 = a, \ b_0 = b, \ a_{n+1} = Aa_n \) and \( b_{n+1} = Ab_n \). It can be proved that \( \{a_n\} \) is nondecreasing, \( \{b_n\} \) is nonincreasing, and
\[
a = a_0 \leq a_1 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_1 \leq b_0 = b.
\]

Note that \( a_n \) is the solution to
\[
\begin{cases}
x'(t) = M a_{n-1}(t) + F(t, a_{n-1}(t), \int_0^t q(t, s, a_{n-1}(s))ds), & t \in [0, T], \\
x(t) = \phi(t), & t \in [-r, 0], \\
x(0) = I_k(a_{n-1}(t_k)), & k = 1, 2, \ldots, m,
\end{cases}
\]
and \( b_n \) is the solution to
\[
\begin{cases}
x'(t) = M b_{n-1}(t) + F(t, b_{n-1}(t), \int_0^t q(t, s, b_{n-1}(s))ds), & t \in [0, T], \\
x(t) = \phi(t), & t \in [-r, 0], \\
x(0) = I_k(b_{n-1}(t_k)), & k = 1, 2, \ldots, m.
\end{cases}
\]

\[\blacksquare\]

**Theorem 3.7.** Let \( a, b \in \Omega' \) be, respectively, \( \leq_n \)-lower and \( \leq_n \)-upper solutions for equations (2) with \( a \leq_n b \) on \([-r, T]\). By condition of Lemma 3.6 and hypotheses (F1)-(F7), there exist monotone sequences \( \{a_n\} \uparrow \rho, \ \{b_n\} \downarrow \gamma \) in \( \Omega' \), where \( a_0 = a, \ b_0 = b, \) and \( \rho, \gamma \) are the extremal solutions to equations (2) in the fuzzy functional interval
\[\{a, b\} := \{x \in \Omega' : a \leq_n x \leq_n b \ \text{on} \ [-r, T]\}.
\]

**Proof.** We prove that \( \{a_n\} \) and \( \{b_n\} \) are uniformly equicontinuous in \( C([-r, T], (E_\alpha)^n) \).
In consequence, there exist convergent subsequences \( \{a_n\} \to \rho, \ \{b_n\} \to \gamma \) in \( C([-r, T], (E_\alpha)^n) \), hence, by monotonicity \( \{a_n\} \to \rho, \ \{b_n\} \to \gamma \) in \( C([-r, T], (E_\alpha)^n) \), that is, \( \{a_n\} \to \rho, \ \{b_n\} \to \gamma \) in \( \Omega' \). In order to apply given in Lemma 3.5, we have to prove that \( \{a_n\}, \ \{b_n\} \) are equicontinuous in \( C([-r, T], (E_\alpha)^n) \).

Indeed, for each \( n \in N \), using the integral representation of \( a_n \) in \([-r, T]\), we obtain
\[
a_n(t) = S(t) a_0(0) + \int_0^t S(t-s) F(s, (\tilde{a}_{n-1})_s(s), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_\tau(\tau))d\tau)ds + \sum_{0 < t_k < t} S(t - t_k) I_k(a_{n-1}(t_k)), \ t \in [0, T],
\]
\[
a_n(t) = \phi(t), \ t \in [-r, 0].
\]

The set \( \{a_n : n \in N\} \) is uniformly equicontinuous in the variable \( t, \alpha \). For \( t + \theta \in [-r, 0] \), then for \( \alpha, \beta \in [0, 1] \), given \( \varepsilon > 0 \) there exists \( \delta_1 > 0 \) such that \(|\alpha - \beta| < \delta_1\)
\[
d_L((a_n)_\alpha(\phi))^{\alpha}, ((a_n)_\beta(\phi))^{\beta} < \varepsilon.
\]
By hypotheses (F5)-(F7), for $t + \theta \in [0, T]$, then for $\alpha, \beta \in [0, 1]$, given $\varepsilon > 0$ there exist $\delta_2, \delta_3, \delta_4 > 0$ such that $|\alpha - \beta| < \min\{\delta_2, \delta_3, \delta_4\}$,

$$
d_L\left(\left[\left((a_n, t)(\phi)^\alpha\right), \left((a_n, t)(\phi)^\beta\right)\right]\right)
= d_L\left(\left[S(t + \theta)(0) + \int_0^{t+\theta} S(t + \theta - s)F\left(s, (\tilde{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_s(\phi))d\tau\right)ds \right.ight.
+ \left. \left. \sum_{0 < t_k < t + \theta} S(t + \theta - t_k)I_k((a_{n-1})(t_k))\right]\right)\alpha,
$$

$$
\leq d_L\left(\left[S(t + \theta)(0)\right]^\alpha, \left[S(t + \theta)(0)\right]^\beta\right)
+ \int_0^{t+\theta} d_L\left(\left[S(t + \theta - s)F\left(s, (\tilde{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_s(\phi))d\tau\right)\right]^\alpha, \left[S(t + \theta - s)F\left(s, (\tilde{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_s(\phi))d\tau\right)\right]^\beta\right)ds
+ d_L\left(\left[\sum_{0 < t_k < t + \theta} S(t + \theta - t_k)I_k((a_{n-1})(t_k))\right]^\alpha, \left[\sum_{0 < t_k < t + \theta} S(t + \theta - t_k)I_k((a_{n-1})(t_k))\right]^\beta\right)
\leq c_{d_L} \left(\left[\phi(0)\right]^\alpha, \left[\phi(0)\right]^\beta\right)
+ c\int_0^{t+\theta} d_L\left(\left[F\left(s, (\tilde{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_s(\phi))d\tau\right)\right]^\alpha, \left[F\left(s, (\tilde{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\tilde{a}_{n-1})_s(\phi))d\tau\right)\right]^\beta\right)ds
+ c_{d_L} \left[\left[\sum_{0 < t_k < t + \theta} I_k((a_{n-1})(t_k))\right]^\alpha, \left[\sum_{0 < t_k < t + \theta} I_k((a_{n-1})(t_k))\right]^\beta\right]
\leq c_{d_L} + c\int_0^{t+\theta} \left(\varepsilon_{d_L} + \int_0^s \varepsilon_{d_L} ds\right)ds + c_{d_L} \varepsilon_{d_L}.\]

$$
H_1(\tilde{a}_n t, (a_n) t) = \sup_{t \in [-r, 0]} D_L((a_n) t, (a_n) t)
$$
Thus the set \( \{ \{ a_n \} : n \in \mathbb{N} \} \) is uniformly equicontinuous in the variable \( \alpha \in [0, 1] \).

And for \( k = 1, 2, \cdots, m \),

\[
d_L \left( [ (a_n)_r(\phi) ]^\alpha, [ (a_n)_{r-\epsilon}(\phi) ]^\alpha \right)
= d_L \left( \left[ S(t + \theta)\phi(0) \right] \right.
+ \int_0^{t+\theta} S(t + \theta - s)F \left( s, (\bar{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\bar{a}_{n-1})_{\tau-\epsilon}(\phi))d\tau \right) ds
+ \sum_{0 < t_k < t+\theta} S(t + \theta - t_k)I_k([a_{n-1}(t_k)])^\alpha,

\[
+ \int_0^{t-t' + \theta} S(t - t' + \theta - s)F \left( s, (\bar{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\bar{a}_{n-1})_{\tau-\epsilon}(\phi))d\tau \right) ds
+ \sum_{0 < t_k < t-t' + \theta} S(t - t' + \theta - t_k)I_k([a_{n-1}((t - t'_k)])^\alpha)
\]
\[
\leq d_L \left( \int_0^{t+\theta} \left[ S(t + \theta - s)F \left( s, (\bar{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\bar{a}_{n-1})_{\tau-\epsilon}(\phi))d\tau \right) \right]^\alpha ds,
\int_0^{t-t' + \theta} \left[ S(t - t' + \theta - s)F \left( s, (\bar{a}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\bar{a}_{n-1})_{\tau-\epsilon}(\phi))d\tau \right) \right]^\alpha ds
\right)
+ d_L \left( \sum_{0 < t_k < t+\theta} S(t + \theta - t_k)I_k([a_{n-1}(t_k)])^\alpha, \right)
\leq c\|d_L\| \left( \int_0^{t+\theta} [ (a_{n-1})_s(\phi) ]^\alpha ds, \int_0^{t-t' + \theta} [ (a_{n-1})_{s-t'}(\phi) ]^\alpha ds \right)
\leq c_k\|d_L\| \left( \int_0^{t+\theta} [ (a_{n-1})_r(\phi) ]^\alpha d\tau, \int_0^{t-t' + \theta} [ (a_{n-1})_{r-t'}(\phi) ]^\alpha d\tau \right) + c\|d_L\| \left( [ (a_{n-1})((t_k)) ]^\alpha, [ (a_{n-1})((t - t'_k)]^\alpha \right).
Then by hypotheses (F1)-(F3), provided we take \( t > t' > 0 \), then we have

\[
H_1((a_n)_t, (a_n)_{t-t'}) = \sup_{\theta \in [-r,0]} D_L((a_n)_t(\phi), (a_n)_{t-t'}(\phi)) \\
= \sup_{\theta \in [-r,0]} \sup_{\alpha \leq 1} D_L((a_n)_t(\phi))^\alpha, ((a_n)_{t-t'}(\phi))^\alpha \\
\leq ch \sup_{\theta \in [-r,0]} \sup_{\alpha \leq 1} D_L(\int_0^{t+\theta} (a_n(t))(\phi)^\alpha ds, \int_0^{t-t'+\theta} ((a_n(t))(\phi)^\alpha ds) \\
+ckp \sup_{\theta \in [-r,0]} \sup_{\alpha \leq 1} D_L(\int_0^{t+\theta} \int_0^s (a_n(t))(\phi)^\alpha ds, \int_0^{t-t'+\theta} \int_0^s ((a_n(t))(\phi)^\alpha ds) \\
\leq chT \sup_{\theta \in [-r,0]} D_L((a_n(t))(\phi), ((a_n(t))(t-t'))^\alpha) \\
+ckpT^2 \sup_{\theta \in [-r,0]} D_L((a_n(t))(\phi), ((a_n(t))(t-t'))^\alpha) \\
\leq c(d + hT + kpT^2)H_1((a_n)_t(\phi), (a_n)_{t-t'}(\phi)).
\]

Hence \( H_1((a_n)_t(\phi), (a_n)_{t-t'}(\phi)) \to 0 \) as \( t' \to 0 \) the set \( \{a_n\} : n \in N \} \) is uniformly equicontinuous in the variable \( t \in [-r, T] \). The case of equicontinuity from the right is similar.

In consequence, \( \{a_n\} : n \in N \} \) is uniformly equicontinuous in the variable \( \alpha \in [0,1], t \in [-r, T] \). This proves that \( \{a_n\} : n \in N \} \) is uniformly equicontinuous in \( C([-r, T], (E_N)^\alpha)) \). And proceeding similarly, and also for \( \{b_n\} \). Hence \( \{a_n\}, \{b_n\} \in B. \) Next, we have to prove that \( \rho, \gamma \) are solutions to (2). To check that \( \rho \) is a solution to (2), we prove that
\[ \rho_t(\phi) = S(t)\phi(0) + \int_0^t S(t-s)F(s, \hat{\rho}_s(\phi), \int_0^s q(s, \tau, \hat{\rho}_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t} S(t-t_k)I_k(\rho(t_k)), \ t \in [0, T], \]

where \( \rho(t) = \phi(t), \ t \in [-r, 0], \)
and \( \rho \) is a solution to (2). The previous problem have solution since \( \rho \in \Omega', \) and \( a_n \leq a \leq b \) on \( \Omega', \) for every \( n, \) hence, \( a \leq n \rho \leq b \) on \( \Omega'. \) For \( k = 1, 2, \cdots, m, \) we prove that the limit of the following expression is zero as \( n \) tends to \( +\infty. \)

\[
d_L\left( [(a_n)_t(\phi)]^\alpha, [\rho_t(\phi)]^\alpha \right)
= d_L\left( S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)F(s, (\hat{\alpha}_{n-1})_s(\phi), \int_0^s q(s, \tau, (\hat{\alpha}_{n-1})_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))]^\alpha, \right.
\]

\[
\left. \left[ S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)F(s, \hat{\rho}_s(\phi), \int_0^s q(s, \tau, \hat{\rho}_\tau(\phi))d\tau)ds + \sum_{0 < t_k < t+\theta} S(t+\theta-t_k)I_k(\rho(t_k))]^\alpha \right) \right]
\leq ch \int_0^{t+\theta} d_L\left( [(a_{n-1})_s(\phi)]^\alpha [\rho_s(\phi)]^\alpha \right)ds
+ckp \int_0^{t+\theta} \int_0^s d_L\left( [(a_{n-1})_\tau(\phi)]^\alpha [\rho_\tau(\phi)]^\alpha \right)d\tau ds
+ckp \int_0^{t+\theta} d_L\left( [(a_{n-1})(t_k)]^\alpha [\rho(t_k)]^\alpha \right).\]

Hence

\[
H_1((a_n)_t, \rho_t) = \sup_{\theta \in [-r, 0]} D_L((a_n)_t(\phi), \rho_t(\phi))
= \sup_{\theta \in [-r, 0]} \sup_{0 < \alpha \leq 1} d_L\left( [(a_n)_t(\phi)]^\alpha, [\rho_t(\phi)]^\alpha \right)
\leq ch \sup_{\theta \in [-r, 0]} \sup_{0 < \alpha \leq 1} \int_0^{t+\theta} d_L\left( [(a_{n-1})_s(\phi)]^\alpha [\rho_s(\phi)]^\alpha \right)ds
\]
For a positive constant \( M \),

if \( x \rho \) that \( H \) is a solution to (2) such that

\[
\begin{align*}
&+ c \sup_{\theta \in [-T,0]} \sup_{0 < \alpha \leq 1} \int_0^{t+\theta} \int_0^s \mathcal{L}\left([\{a_{n-1}\}t(\phi)]^\alpha, [\rho(t)]^\alpha\right) dr ds \\
&+ c \sup_{\theta \in [-T,0]} \sup_{0 < \alpha \leq 1} \mathcal{L}\left([\{a_{n-1}\}t(\phi), \rho(t)\right) \\
&\leq c(1 + hT + k T^2) \mathcal{L}\left((a_{n-1})_{t(\phi), \rho(t)}\right) \\
&\leq c(d + hT + k T^2) H_1 \left((a_{n-1})_{t, \rho}\right).
\end{align*}
\]

Therefore \( H_1((a_n)_t, \rho_t) \to 0 \) as \( n \to +\infty \).

Using that \( I_k \) and \( F \) are continuous convergence of \( a_{n-1} \) towards \( \rho \), we obtain that \( \rho \) is a solution to (2). For function \( \gamma \), we follow a similar procedure. Finally, if \( x \) is a solution to (2) such that \( a \leq n x \leq b \), using that \( A \) is nondecreasing, we obtain

\[
a_n = A^n a \leq n A^n x = x \leq n A^n b = b_n,
\]

then, by Lemma 3.4,

\[
\rho \leq n x \leq n \gamma.
\]

In conclusion, \( x \) exists between \( \rho \) and \( \gamma \). \( \square \)

4. Example

We consider the following two one-dimensional impulsive delay fuzzy integrodifferential equations

\[
\begin{align*}
\left\{ \frac{dx_i(t)}{dt} = f_i \left( t, (x_i)_t, \int_0^t q_i(t, s, (x_i)_s) ds \right), \ t \in J, \\
x_i(t) = \phi_i(t), \ t \in J, \\
x_i(t_{i+1}^-) = I_k(x_i(t_k)), \ t \neq t_k, \ k = 1, 2, \ldots, m, \ i = 1, 2,
\end{align*}
\]

where \( T > 0, J = [0,T], 0 < t_1 < \cdots < t_m < t_{m+1} = T, E_N, i = 1, 2, \) is the set of all upper semi-continuously convex fuzzy numbers on \( R \) with \( E_N^i \neq E_N^j \ (i \neq j) \), \( f_i : J \times E_N^i \times E_N^i \to E_N \) and \( q_i : J \times J \times E_N^i \to E_N \) are regular continuous fuzzy function, \( (x_i)_t = x_i(t + \theta), \ \theta \in [-r,0], \phi_i \in C([-r,0], E_N^i) \) is initial function and \( I_k \in C(E_N^i, E_N^i) \) are bounded functions.

Let for \( i = 1, 2, \)

\[
f_i \left( t, (x_i)_t, \int_0^t q_i(t, s, (x_i)_s) ds \right) = \tilde{f}(x_i)_t^2 + \int_0^t (t-s)(x_i)_s ds, \ t \in [0,T].
\]

For a positive constant \( M \), let

\[
\frac{dx_i(t)}{dt} = M(x_i)_t + \tilde{f}(x_i)_t^2 + \int_0^t (t-s)(x_i)_s ds - M(x_i)_t.
\]
Put
\[
F(t, x_i(t), \int_0^t q_i(t, s, x_i(s))ds) = \tilde{2t}(x_i(t))^2 + \int_0^t (t - s)(x_i(s))ds - M(x_i(t),
\]
\[x_i(t) = \phi_i(t), \quad i = 1, 2, \quad t \in [-r, 0],
\]
\[
\Delta x_i(t_k) = x_i(t_k^+ - x_i(t_k^-)
\]
is impulsive effect at \( t = t_k(k = 1, 2, \ldots, m) \). Let
\[
F(t, x_i, \int_0^t q(t, s, x_s)ds)
\]
\[
= \left( F_1(t, x_1(t), \int_0^t q_1(t, s, x_1(s))ds), F_2(t, x_2(t), \int_0^t q_2(t, s, x_2(s))ds) \right)
\]
\[
= \left( \tilde{2t}(x_1(t))^2 + \int_0^t (t - s)(x_1(s))ds - M(x_1(t)), \tilde{2t}(x_2(t))^2 + \int_0^t (t - s)(x_2(s))ds - M(x_2(t)) \right),
\]
\[
\Delta x_i(t_k) = \Delta x_i(t_k), \Delta x_2(t_k), \quad t \neq t_k, \quad k = 1, 2, \ldots, m,
\]
\[
I_k(x_i(t_k)) = (I_k(x_i(t_k)), I_k(x(x_i(t_k)))
\]
\[
= (x_1(t_k^+), x_2(t_k^+ - x_2(t_k^-))
\]
\[
= (x_1(t_k^+), x_2(t_k^+), x_2(t_k^-), x_2(t_k^-), \quad t \neq t_k, \quad k = 1, 2, \ldots, m.
\]

We consider the following equations
\[
\begin{cases}
\begin{aligned}
\frac{dx(t)}{dt} &= Mx(t) + F(t, x(t), \int_0^t q(t, s, x_s)ds), \quad t \in [0, T],
x(t) = \phi(t), \quad t \in [-r, 0],
x(t_k^+) = I_k(x_i(t_k))), \quad t \neq t_k, \quad k = 1, 2, \ldots, m.
\end{aligned}
\end{cases}
\]

The \( \alpha \)-level set of fuzzy numbers are the following:
\[
[0]_\alpha = [\alpha - 1, 1 - \alpha], \quad [1]_\alpha = [\alpha + 1, 2 - \alpha] \quad \text{for all} \ \alpha \in [0, 1], \quad M = 1.
\]

Then \( \alpha \)-level set of \( F(t, x(t), \int_0^t q(t, s, x_s)ds) \) is
\[
\left[ F(t, x(t), \int_0^t q(t, s, x_s)ds) \right]^\alpha
\]
\[
= \left[ \tilde{2t}(x_1(t))^2 + \int_0^t (t - s)(x_1(s))ds - (x_1(t))^2 \right]^\alpha
\]
\[
\times \left[ \tilde{2t}(x_2(t))^2 + \int_0^t (t - s)(x_2(s))ds - (x_2(t))^2 \right]^\alpha
\]
\[
= \left[ \tilde{2t}(x_1(t))^2 + \int_0^t (t - s)(x_1(s))ds - (x_1(t))^2 \right]^\alpha
\]
\[
\times \left[ \tilde{2t}(x_2(t))^2 + \int_0^t (t - s)(x_2(s))ds - (x_2(t))^2 \right]^\alpha
\]
\[
\times \left[ \tilde{2t}(x_1(t))^2 + \int_0^t (t - s)(x_1(s))^\alpha ds - [(x_1(t))^\alpha] \right]^\alpha
\]
\[
\times \left[ \tilde{2t}(x_2(t))^2 + \int_0^t (t - s)(x_2(s))^\alpha ds - [(x_2(t))^\alpha] \right]^\alpha
\]
where $h$ and $k$ satisfy the hypothesis (F1). Since $I_k$ is a bounded function, we know that the hypothesis (F2) holds. Choose $T$ such that $T < (1 - cd)/ch$. Then all conditions stated in Theorem 3.3 are satisfied, so the equations (6) have a
unique fuzzy solution. Since \( \phi \) is a continuous function on \([-r, T]\), hypothesis (F5) satisfied. Next, we show that the hypothesis (F6) satisfies. For \( \alpha, \beta \in [0, 1] \) and given \( \varepsilon = \frac{1}{3\varepsilon T} \delta_3 > 0 \), there exist

\[
\delta_3 = \frac{\varepsilon}{(3T|\alpha| + (\beta_1 + \frac{T^2}{T} + 1)} > 0
\]

such that

\[
|\alpha - \beta| < \delta_3, \quad d_L([x_1]^{\alpha}, [x_1]^{\beta}) < \varepsilon,
\]

then

\[
d_L \left( \left[ F \left( t, \bar{x}_t, \int_0^t q(t, s, \bar{x}_s) ds \right) \right]^\alpha, \left[ F \left( t, \bar{x}_t, \int_0^t q(t, s, \bar{x}_s) ds \right) \right]^\beta \right)
\]

\[
= d_L \left( \prod_{i=1}^2 \left[ (\alpha + 1)(x_i^\alpha)_t^2 + \int_0^t (t - s)(x_i^\alpha)_s ds - (x_i^\beta)_t, (\beta + 1)(x_i^\beta)_t^2 + \int_0^t (t - s)(x_i^\beta)_s ds - (x_i^\alpha)_t \right] \right)
\]

\[
\leq t \max_{1 \leq i \leq 2} \{(\alpha + 1)(x_i^\alpha)_t^2 - (\beta + 1)(x_i^\beta)_t^2, (\beta - \alpha)(x_i^\alpha)_t^2 - (3 - \beta)(x_i^\beta)_t^2\}
\]

\[
+ \int_0^t (t - s) \max_{1 \leq i \leq 2} \{(x_i^\alpha)_s - (x_i^\beta)_s\}ds
\]

\[
+ \max_{1 \leq i \leq 2} \{(x_i^\alpha)_t - (x_i^\beta)_t\}
\]

\[
\leq 3T \max_{1 \leq i \leq 2} \max \{(x_i^\alpha)_t, (x_i^\beta)_t, (x_i^\alpha)_t, (x_i^\alpha)_t\}
\]

\[
+ \left( \frac{T^2}{2} + 1 \right) \max_{1 \leq i \leq 2} \{(x_i^\alpha)_t - (x_i^\beta)_t\}
\]

\[
\leq 3T \left( x_i^\alpha \right)_t + (x_i^\beta)_t [d_L([x_1]^{\alpha}, [x_1]^{\beta}) + \left( \frac{T^2}{2} + 1 \right) d_L([x_1]^{\alpha}, [x_1]^{\beta})
\]

\[
= \left( 3T \left( x_i^\alpha \right)_t + (x_i^\beta)_t \right) + \left( \frac{T^2}{2} + 1 \right) d_L([x_1]^{\alpha}, [x_1]^{\beta})
\]

\[
\leq \left( 3T \left( x_i^\alpha \right)_t + (x_i^\beta)_t \right) + \left( \frac{T^2}{2} + 1 \right) \frac{1}{3T} \delta_3
\]

\[
\leq \frac{\varepsilon}{3T}.
\]

Then all conditions stated in Theorem 3.7 are satisfied, so the problem (5) has a unique extremal solution.
References


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