

L -ORDERED FUZZIFYING CONVERGENCE SPACES

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ABSTRACT. Based on a complete Heyting algebra, we modify the definition of lattice-valued fuzzifying convergence space using fuzzy inclusion order and construct in this way a Cartesian-closed category, called the category of L -ordered fuzzifying convergence spaces, in which the category of L -fuzzifying topological spaces can be embedded. In addition, two new categories are introduced, which are called the category of principal L -ordered fuzzifying convergence spaces and that of topological L -ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of L -fuzzifying neighborhood spaces and that of L -fuzzifying topological spaces respectively.

1. Introduction

Convergence structures are more general than topological structures. If a convergence structure additionally satisfies proper conditions, it is equivalent to a topological structure. Lowen [12] constructed convergence systems using prefilters, through which Min [13] proposed fuzzy limit structures. Xu [14] proved that topological L -fuzzifying convergence structures and L -fuzzifying topologies [17] are equivalent, where classical filters play a crucial role. By stratified L -filters [7], Jäger [8] introduced stratified L -fuzzy convergence spaces in the many-valued case. The category of these spaces was developed to a significant extent in the recent years [1,2,4,5,9-11,14,15].

In 2009, Yao [16] defined L -fuzzifying convergence spaces, and showed the category of L -fuzzifying topological spaces [17] could be embedded in the category of L -fuzzifying convergence spaces as a reflective subcategory and the latter is Cartesian-closed. L -fuzzifying convergence spaces were based on L -filters of ordinary subsets.

This paper can be seen as a further step towards [16]. It proposes a new lattice-valued fuzzifying convergence structure, called L -ordered fuzzifying convergence structure, which is compatible with the fuzzy inclusion order of L -filters of ordinary subsets. The category of L -fuzzifying topological spaces [17] can be embedded in the resulting category. As a matter of fact, it is easier for a bigger category to be Cartesian-closed, and it makes sense to establish a smaller Cartesian-closed category. Note that the category of L -ordered fuzzifying convergence spaces is

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“smaller” than that of L -fuzzifying convergence spaces [16], and it is Cartesian-closed. In addition, two new categories are introduced, which are called the category of principal L -ordered fuzzifying convergence spaces and that of topological L -ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of L -fuzzifying neighborhood spaces and that of L -fuzzifying topological spaces respectively.

2. Preliminaries

Let (L, \vee, \wedge) be a complete lattice. If the finite meets are distributive over arbitrary joins, i.e. for all $a, b_i \in L, (i \in J)$

$$a \wedge \left(\bigvee_{i \in J} b_i \right) = \bigvee_{i \in J} (a \wedge b_i),$$

L is called a complete Heyting algebra. For L , we define an implication operator $\rightarrow: L \times L \rightarrow L$ as follows:

$$\forall a, b \in L, a \rightarrow b = \bigvee \{c \in L \mid a \wedge c \leq b\}.$$

Then it is the right adjoint for \wedge , i.e.,

$$\forall a, b, c \in L, a \wedge c \leq b \Leftrightarrow c \leq a \rightarrow b.$$

Theorem 2.1. [7] *Let L be a complete Heyting algebra. For all $a, b, c, d, a_i, b_i \in L, (i \in J)$, the following holds:*

- (H1) $a \leq (b \rightarrow c) \Leftrightarrow a \wedge b \leq c$, and $a \leq b \Leftrightarrow (a \rightarrow b) = 1$,
- (H2) $a \rightarrow (\bigwedge_{i \in J} b_i) = \bigwedge_{i \in J} (a \rightarrow b_i)$, $(\bigvee_{i \in J} b_i) \rightarrow a = \bigwedge_{i \in J} (b_i \rightarrow a)$,
- (H3) $(b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$, $(a \rightarrow c) \wedge (b \rightarrow d) \leq (a \wedge b) \rightarrow (c \wedge d)$,
- (H4) $a \rightarrow b \geq b$, $a \leq (a \rightarrow b) \rightarrow b$,
- (H5) $a \wedge b = a \wedge (a \rightarrow b)$, therefore, $b = 1 \rightarrow b$,
- (H6) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$,
- (H7) $\bigwedge_{i \in J} (a_i \rightarrow b_i) \leq (\bigwedge_{i \in J} a_i) \rightarrow (\bigwedge_{i \in J} b_i)$.

In what follows, we consider X a nonempty set and L a complete Heyting algebra unless otherwise stated.

For a given set X , L^X denotes the set of all L -subsets on X . Define a binary mapping $S(-, -): L^X \times L^X \rightarrow L$ by $S(U, V) = \bigwedge_{x \in X} (U(x) \rightarrow V(x))$ for each pair $(U, V) \in L^X \times L^X$.

Definition 2.2. [6] A map $\mathcal{F}: 2^X \rightarrow L$ is called an L -filter of ordinary subsets of X if it satisfies $\forall x \in X, A, B \in 2^X$,

- (F1) $\mathcal{F}(\emptyset) = 0, \mathcal{F}(X) = 1$,
- (F2) $A \subseteq B \Rightarrow \mathcal{F}(A) \leq \mathcal{F}(B)$,
- (F3) $\mathcal{F}(A \cap B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$.

The family of all L -filters of ordinary subsets on X will be denoted by $\mathcal{F}_L(X)$. An order on $\mathcal{F}_L(X)$ is defined as follows: $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall U \in 2^X, \mathcal{F}(U) \leq \mathcal{G}(U)$.

For every $x \in X$, $[x] \in \mathcal{F}_L(X)$ is defined by $\forall A \in 2^X$,

$$[x](A) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Let \mathcal{F} be a filter of ordinary subsets on X and $f : X \rightarrow Y$ be a mapping. Then the mapping $f \Rightarrow (\mathcal{F}) : 2^Y \rightarrow L$, where $\forall B \in 2^Y, f \Rightarrow (\mathcal{F})(B) = \mathcal{F}(f \leftarrow (B))$, is an L -filter of ordinary subsets on Y and is called the image of \mathcal{F} under f .

For every $\mathcal{F} \in \mathcal{F}_L(X), \mathcal{G} \in \mathcal{F}_L(Y), \mathcal{F} \times \mathcal{G} \in \mathcal{F}_L(X \times Y)$ is defined as follows: $\forall C \in 2^{X \times Y}, (\mathcal{F} \times \mathcal{G})(C) = \bigvee_{A \times B \subseteq C} \mathcal{F}(A) \wedge \mathcal{G}(B)$.

Definition 2.3. [18] An L -fuzzifying neighborhood structure on a set X is a family of functions $N = \{N_x : 2^X \rightarrow L \mid x \in X\}$ with the following conditions: For all $x \in X, U, V \in 2^X$,

$$(LN1) \quad N_x(X) = 1,$$

$$(LN2) \quad N_x(U) > 0 \text{ implies } x \in U,$$

$$(LN3) \quad N_x(U \cap V) = N_x(U) \wedge N_x(V).$$

The pair (X, N) is called an L -fuzzifying neighborhood space, and it will be called topological if it satisfies moreover: For all $x \in X, U \in 2^X$,

$$(LN4) \quad N_x(U) = \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N_y(V).$$

A continuous function between L -fuzzifying neighborhood spaces (X, N^1) and (Y, N^2) is a map $f : X \rightarrow Y$ such that for all $x \in X, U \in 2^Y, N_x^1(f \leftarrow (U)) \geq N_{f(x)}^2(U)$.

Let L -NGH denote the category of L -fuzzifying neighborhood spaces with continuous maps, and L -TNGH the full subcategory of L -NGH consisting of topological L -fuzzifying neighborhood spaces.

Definition 2.4. [17] An L -fuzzifying topology on X is a function $\tau : 2^X \rightarrow L$ which satisfies

$$(FO1) \quad \tau(\emptyset) = \tau(X) = 1,$$

$$(FO2) \quad \tau(A \cap B) \geq \tau(A) \wedge \tau(B),$$

$$(FO3) \quad \tau(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau(A_j).$$

For an L -fuzzifying topology τ on X , the pair (X, τ) is called an L -fuzzifying topological space. A map $f : X \rightarrow Y$ is called continuous with respect to the given L -fuzzifying topological spaces (X, τ_1) and (Y, τ_2) iff $\forall B \in 2^Y, \tau_1(f \leftarrow (B)) \geq \tau_2(B)$. The category of L -fuzzifying topological spaces with continuous maps as morphisms will be denoted by L -FYS.

It was proved in [20] that for any completely distributive lattice L , topological L -fuzzifying neighborhood systems and L -fuzzifying topologies are conceptually equivalent with transferring process $N_x(U) = \bigvee_{x \in V \subseteq U} \tau(V)$ and $\tau(U) = \bigwedge_{x \in U} N_x(U)$.

Theorem 2.5. [19] Let $\varphi : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a mapping. If L is a completely distributive lattice, then φ is continuous iff $N_x^{\tau_1}(\varphi^{\leftarrow}(U)) \geq N_{\varphi(x)}^{\tau_2}(U), \forall x \in X, U \in 2^Y$.

3. L -ordered Fuzzifying Convergence Structure

In [16], the author developed lattice-valued convergence structure $\lim : \mathcal{F}_L(X) \rightarrow L^X$ as follows:

Definition 3.1. [16] A mapping $\lim : \mathcal{F}_L(X) \rightarrow L^X$, subject to the conditions

$$(LY1) \quad \forall x \in X, \limx = 1,$$

$$(LY2) \quad \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \mathcal{F} \leq \mathcal{G} \Rightarrow \forall x \in X, \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x),$$

is called an L -fuzzifying convergence structure on X , and (X, \lim) an L -fuzzifying convergence space.

The set of all L -fuzzifying convergence structures on X is denoted by $\lim_{ly}(X)$. An order on $\lim_{ly}(X)$ can be defined by $\lim_1 \leq \lim_2$ iff for all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x)$.

In Definition 3.1, the L -filters in the axiom (LY2) are in nature L -sets on 2^X . We use the method in [3] and define an L -partial order $S_F(-, -)$ on $\mathcal{F}_L(X)$ as follows: $S_F(-, -) : \mathcal{F}_L(X) \times \mathcal{F}_L(X) \rightarrow L$

$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), S_F(\mathcal{F}, \mathcal{G}) = \bigwedge_{U \in 2^X} (\mathcal{F}(U) \rightarrow \mathcal{G}(U)).$$

In this case, we can redefine the axiom (LY2) in Definition 3.1, proposing the following new lattice-valued convergence structure.

Definition 3.2. An L -fuzzifying convergence structure $\lim : \mathcal{F}_L(X) \rightarrow L^X$, satisfying the following condition:

$$(OLY2) \quad \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), S_F(\mathcal{F}, \mathcal{G}) \leq S(\lim \mathcal{F}, \lim \mathcal{G}),$$

is called an L -ordered fuzzifying convergence structure, and the pair (X, \lim) an L -ordered fuzzifying convergence space.

A function $\varphi : (X, \lim^X) \rightarrow (Y, \lim^Y)$, $(X, \lim^X), (Y, \lim^Y)$ L -ordered fuzzifying convergence spaces, is called continuous iff for all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim^Y \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)) \leq \lim^X \mathcal{F}(x)$.

We do not go into details here, but only remark that (OLY2) implies (LY2).

The next example shows there exists an L -fuzzifying convergence structure \lim which is not an L -ordered fuzzifying convergence structure.

Example 3.3. Let $X = \{x, y\}$, $L = \{0, \alpha, 1\}$ be a chain. Define a map $\lim : \mathcal{F}_L(X) \rightarrow L^X, \forall \mathcal{F} \in \mathcal{F}_L(X), z \in X$,

$$\lim \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} \geq [z], \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that \lim is an L -fuzzifying convergence structure. Define a mapping $\mathcal{F}^* : 2^X \rightarrow L$ as follows: $\forall A \in 2^X$,

$$\mathcal{F}^*(A) = \begin{cases} 1, & A = X, \\ \alpha, & A = \{x\}, \\ 0, & A = \{y\} \text{ or } A = \emptyset. \end{cases}$$

It can be verified that \mathcal{F}^* is an L -filter of ordinary subsets on X . Then

$$\begin{aligned} S_F([x], \mathcal{F}^*) &= \bigwedge_{A \in 2^X} ([x](A) \rightarrow \mathcal{F}^*(A)) \\ &= ([x](\emptyset) \rightarrow \mathcal{F}^*(\emptyset)) \wedge ([x](\{x\}) \rightarrow \mathcal{F}^*(\{x\})) \\ &\quad \wedge ([x](\{y\}) \rightarrow \mathcal{F}^*(\{y\})) \wedge ([x](X) \rightarrow \mathcal{F}^*(X)) \\ &= 1 \wedge \alpha \wedge 1 \wedge 1 \\ &= \alpha \end{aligned}$$

And

$$\begin{aligned} S(\lim[x], \lim \mathcal{F}^*) &= \bigwedge_{z \in X} (\lim[x](z) \rightarrow \lim \mathcal{F}^*(z)) \\ &= (\limx \rightarrow \lim \mathcal{F}^*(x)) \wedge (\lim[x](y) \rightarrow \lim \mathcal{F}^*(y)) \\ &= (1 \rightarrow 0) \wedge (0 \rightarrow 0) \\ &= 0 \end{aligned}$$

We can see that $S_F([x], \mathcal{F}^*) \not\leq S(\lim[x], \lim \mathcal{F}^*)$, hence \lim is not an L -ordered fuzzifying convergence structure.

Example 3.4. Let $(X, \tau) \in L\text{-FYS}$ and define a mapping $\lim_\tau: \mathcal{F}_L(X) \rightarrow L^X$, $\forall \mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim_\tau \mathcal{F}(x) = S_F(N_\tau^x, \mathcal{F})$. Here, the L -fuzzifying neighborhood system N_τ^x of $x \in X$ is defined by $N_\tau^x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$. Then \lim_τ is an L -ordered fuzzifying convergence structure.

From Example 3.4, we see that an L -fuzzifying topology can induce an L -ordered fuzzifying convergence structure. The following theorem shows that the induced L -ordered fuzzifying convergence structure from the L -fuzzifying topology can determine the induced L -fuzzifying neighborhood structure from the L -fuzzifying topology. This idea has been presented in [8].

Theorem 3.5. *Let $(X, \tau) \in L\text{-FYS}$. Then the following holds:*

$$N_\tau^x(U) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_\tau \mathcal{F}(x) \rightarrow \mathcal{F}(U)), \forall x \in X, U \in 2^X.$$

Let $L\text{-FYCS}$ [16] denote the category of L -fuzzifying convergence spaces with continuous maps and $L\text{-OFYC}$ the full subcategory of $L\text{-FYCS}$ formed by all L -ordered fuzzifying convergence spaces.

The set of all L -ordered fuzzifying convergence structures on X is denoted by $\lim_{loy}(X)$. An order on $\lim_{loy}(X)$ can be defined by $\lim_1 \leq \lim_2$ iff for all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x)$. For $\lim_{loy}(X)$ here, we immediately

obtain that there are a maximum element and a minimum element in $(\lim_{loy}(X), \leq)$, denoted by \lim_{sm} and \lim_m respectively: $\forall \mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim_{sm} \mathcal{F} = 1_X$; $\lim_m \mathcal{F}(x) = S_F([x], \mathcal{F})$. The supremum element of a family of L -ordered fuzzifying convergence structures $(\lim_j)_{j \in J} \subseteq \lim_{loy}(X)$ is defined by $(\sup_{j \in J} \lim_j) \mathcal{F}(x) = \bigwedge_{j \in J} \lim_j \mathcal{F}(x)$, $\forall \mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$. Obviously, $\sup_{j \in J} \lim_j \in \lim_{loy}(X)$. Therefore, the following proposition holds.

Proposition 3.6. $(\lim_{loy}(X), \leq)$ is a complete lattice.

We will next address the result that the category of L -ordered fuzzifying convergence spaces is a topological category. To this end, we note the following proposition.

Proposition 3.7. The category L -OFYC is a full reflective subcategory in the category L -FYCS.

Proof. Let $(X, \overline{\lim}) \in L$ -FYCS and $E_{\overline{\lim}} = \{ \lim \mid (X, \lim) \in L$ -OFYC, $\lim \leq \overline{\lim} \}$. Note that $E_{\overline{\lim}}$ is not empty because it always contains \lim_{sm} . Then with Proposition 3.6, we can construct an L -ordered fuzzifying convergence structure $\overline{\lim}_* : \mathcal{F}_L(X) \rightarrow L^X$ as follows: For all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\overline{\lim}_* \mathcal{F}(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim \mathcal{F}(x)$. From this, we have

(1) $id_X : (X, \overline{\lim}) \rightarrow (X, \overline{\lim}_*)$ is trivially continuous;

(2) For an L -ordered fuzzifying convergence space (Y, \lim^Y) , if $f : (X, \overline{\lim}) \rightarrow (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \overline{\lim}_*) \rightarrow (Y, \lim^Y)$ is also continuous. We leave the above check to the reader.

From the above facts, we immediately obtain that L -OFYC is a full reflective subcategory in L -FYCS. \square

In [16] Yao proved that the category L -FYCS is topological. By Proposition 3.7, we have the following main result.

Theorem 3.8. The category of L -ordered fuzzifying convergence spaces L -OFYC is topological.

4. The Relations Between Categories of L -FYS and L -OFYC

This section is motivated by reference [8]. In this section, we will resolve the embedding of L -FYS into L -OFYC. By Example 3.4 and Theorem 3.5, we see that L -ordered convergence structures can be induced from L -fuzzifying topologies. Moreover, they are unique. In order to show that L -FYS can be embedded in the category of L -OFYC, the following theorem is necessary.

Theorem 4.1. Let L be a completely distributive lattice. Then the map $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ between two L -fuzzifying topological spaces is continuous iff $f : (X, \lim_{\tau_1}) \rightarrow (Y, \lim_{\tau_2})$ is continuous.

Proof. Suppose that $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is continuous, by Theorem 2.5, we have for all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$,

$$\begin{aligned} \lim_{\tau_2} \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)) &= \bigwedge_{V \in 2^Y} (N_{\tau_2}^{\varphi(x)}(V) \rightarrow \varphi^{\Rightarrow}(\mathcal{F})(V)) \\ &\geq \bigwedge_{V \in 2^Y} (N_{\tau_1}^x(\varphi^{\leftarrow}(V)) \rightarrow \mathcal{F}(\varphi^{\leftarrow}(V))) \\ &\geq \bigwedge_{U \in 2^X} (N_{\tau_1}^x(U) \rightarrow \mathcal{F}(U)) \\ &= \lim_{\tau_1} \mathcal{F}(x). \end{aligned}$$

Hence, $f : (X, \lim_{\tau_1}) \rightarrow (Y, \lim_{\tau_2})$ is continuous.

Conversely, if $f : (X, \lim_{\tau_1}) \rightarrow (Y, \lim_{\tau_2})$ is continuous, by Theorem 3.5, we have $\forall x \in X, U \in 2^Y$,

$$\begin{aligned} N_{\tau_1}^x(\varphi^{\leftarrow}(U)) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_{\tau_1} \mathcal{F}(x) \rightarrow \mathcal{F}(\varphi^{\leftarrow}(U))) \\ &\geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_{\tau_2} (\varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)) \rightarrow (\varphi^{\Rightarrow}(\mathcal{F}))(U))) \\ &\geq \bigwedge_{\mathcal{G} \in \mathcal{F}_L(Y)} (\lim_{\tau_2} \mathcal{G}(\varphi(x)) \rightarrow \mathcal{G}(U)) \\ &= N_{\tau_2}^{\varphi(x)}(U). \end{aligned}$$

Therefore, by Theorem 2.5, $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is continuous. □

As a consequence of the above theorems, we have the following result.

Theorem 4.2. *Let L be a completely distributive lattice. L -FYS can be embedded in the category of L -OFYC.*

In Theorem 3.8 we know that L -OFYC is topological. So, in order to show that it is Cartesian-closed, the following results are necessary. Similar to the definition of product spaces in L -FYCS, it can be shown that there are also product spaces in L -OFYC. We refer the reader to [16]. Here, we only present the main results. Note that for two L -ordered fuzzifying convergence spaces $(X, \lim_X), (Y, \lim_Y)$, let $[X \rightarrow Y]$ denote the set of all continuous maps from (X, \lim_X) to (Y, \lim_Y) .

Lemma 4.3. [16] *Let $g : X \rightarrow Y$ and $\mathcal{G} \in \mathcal{F}_L(X)$, then $g^{\Rightarrow}(\mathcal{G}) \leq ev^{\Rightarrow}([g] \times \mathcal{G})$, where $ev : [X \rightarrow Y] \times X \rightarrow Y$ is the evaluation map.*

Theorem 4.4. *Let $(X, \lim_X), (Y, \lim_Y)$ be L -ordered fuzzifying convergence spaces, then $\lim_{[X \rightarrow Y]} : F_L([X \rightarrow Y]) \rightarrow L^{[X \rightarrow Y]}$, $\forall \mathcal{F} \in F_L([X \rightarrow Y]), \forall f \in [X \rightarrow Y]$, $\lim_{[X \rightarrow Y]} \mathcal{F}(f) = \bigwedge_{(\mathcal{G}, x) \in \mathcal{F}_L(X) \times X} (\lim_X \mathcal{G}(x) \rightarrow \lim_Y ev^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(f(x)))$ is an L -ordered fuzzifying convergence structure on $[X \rightarrow Y]$.*

Proof. For (LY1), $\forall g \in [X \rightarrow Y]$,

$$\begin{aligned} \lim_{[X \rightarrow Y]}g &= \bigwedge_{(\mathcal{G}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{G}(x) \rightarrow \lim_Y (ev^{\Rightarrow}([g] \times \mathcal{G}))(g(x)) \\ &\geq \bigwedge_{(\mathcal{G}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{G}(x) \rightarrow \lim_Y (g^{\Rightarrow}(\mathcal{G}))(g(x)) \\ &= 1. \end{aligned}$$

For (OLY2), $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L([X \rightarrow Y])$,

$$\begin{aligned} &S(\lim_{[X \rightarrow Y]} \mathcal{F}, \lim_{[X \rightarrow Y]} \mathcal{G}) \\ &= \bigwedge_{g \in [X \rightarrow Y]} \left(\left(\bigwedge_{(\mathcal{E}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{E}(x) \rightarrow \lim_Y (ev^{\Rightarrow}(\mathcal{F} \times \mathcal{E}))(g(x)) \right) \right. \\ &\quad \left. \rightarrow \left(\bigwedge_{(\mathcal{H}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{H}(x) \rightarrow \lim_Y (ev^{\Rightarrow}(\mathcal{G} \times \mathcal{H}))(g(x)) \right) \right) \\ &\geq \bigwedge_{g \in [X \rightarrow Y]} \bigwedge_{(\mathcal{H}, x) \in \mathcal{F}_L(X) \times X} \left(\left(\lim_X \mathcal{H}(x) \rightarrow \lim_Y (ev^{\Rightarrow}(\mathcal{F} \times \mathcal{H}))(g(x)) \right) \right. \\ &\quad \left. \rightarrow \left(\lim_X \mathcal{H}(x) \rightarrow \lim_Y (ev^{\Rightarrow}(\mathcal{G} \times \mathcal{H}))(g(x)) \right) \right) \\ &\geq \bigwedge_{\mathcal{H} \in \mathcal{F}_L(X)} S(\lim_Y (ev^{\Rightarrow}(\mathcal{F} \times \mathcal{H})), \lim_Y (ev^{\Rightarrow}(\mathcal{G} \times \mathcal{H}))) \\ &\geq \bigwedge_{\mathcal{H} \in \mathcal{F}_L(X)} S_F(ev^{\Rightarrow}(\mathcal{F} \times \mathcal{H}), ev^{\Rightarrow}(\mathcal{G} \times \mathcal{H})). \end{aligned}$$

$\forall \mathcal{H} \in \mathcal{F}_L(X)$,

$$\begin{aligned} &S_F(ev^{\Rightarrow}(\mathcal{F} \times \mathcal{H}), ev^{\Rightarrow}(\mathcal{G} \times \mathcal{H})) \\ &= \bigwedge_{U \in 2^Y} \left((\mathcal{F} \times \mathcal{H})(ev^{\leftarrow}(U)) \rightarrow (\mathcal{G} \times \mathcal{H})(ev^{\leftarrow}(U)) \right) \\ &= \bigwedge_{U \in 2^Y} \left(\left(\bigvee_{A \times B \subseteq ev^{\leftarrow}(U)} \mathcal{F}(A) \wedge \mathcal{H}(B) \right) \rightarrow \left(\bigvee_{C \times D \subseteq ev^{\leftarrow}(U)} \mathcal{G}(C) \wedge \mathcal{H}(D) \right) \right) \\ &\geq \bigwedge_{U \in 2^Y} \bigwedge_{A \times B \subseteq ev^{\leftarrow}(U)} \left((\mathcal{F}(A) \wedge \mathcal{H}(B)) \rightarrow (\mathcal{G}(A) \wedge \mathcal{H}(B)) \right) \\ &\geq \bigwedge_{U \in 2^Y} \bigwedge_{A \times B \subseteq ev^{\leftarrow}(U)} \left(\mathcal{F}(A) \rightarrow \mathcal{G}(A) \right) \\ &\geq \bigwedge_{C \in 2^{[X \rightarrow Y]}} \left(\mathcal{F}(C) \rightarrow \mathcal{G}(C) \right) \\ &= S_F(\mathcal{F}, \mathcal{G}). \end{aligned}$$

Therefore, the above completes the proof. In other words, $\lim_{[X \rightarrow Y]}$ is an L -ordered fuzzifying convergence structure on $[X \rightarrow Y]$. \square

Remark 4.5. The evaluation map $ev : [X \rightarrow Y] \times X \rightarrow Y$ mentioned above is continuous. Let $f : X \times Y \rightarrow Z$ be a map, $\forall x \in X$, define a map $f_x : Y \rightarrow Z$, $\forall y \in Y$, $f_x(y) = f(x, y)$, $f^* : X \rightarrow Z^Y$, $\forall x \in X$, $f^*(x) = f_x$, and $\varphi : Z^{(X \rightarrow Y)} \rightarrow (Z^Y)^X$, $\forall f \in Z^{(X \rightarrow Y)}$, $\varphi(f) = f^*$. Then it can be proved that

- (1) If $f : (X, \lim_X) \times (Y, \lim_Y) \rightarrow (Z, \lim_Z)$ is continuous, then for each $x \in X$, $f_x : (Y, \lim_Y) \rightarrow (Z, \lim_Z)$ is continuous.
- (2) For all $\mathcal{F} \in \mathcal{F}_L(X)$, $\mathcal{G} \in \mathcal{F}_L(Y)$, $ev \Rightarrow (\varphi(f) \Rightarrow (\mathcal{F} \times \mathcal{G})) = f \Rightarrow (\mathcal{F} \times \mathcal{G})$.
- (3) If $f : X \times Y \rightarrow Z$ is continuous, then $\varphi(f) : X \rightarrow [Y \rightarrow Z]$ is continuous. (We refer to [16] for a detail proof of the above results.)

We collect our findings in the following theorem.

Theorem 4.6. L -OFYC is a Cartesian-closed category.

5. The Relations Between L -fuzzifying Neighborhood Spaces and Principle L -ordered Fuzzifying Convergence Spaces

In this section, we define a subcategory of the category of L -ordered fuzzifying convergence spaces: the category of principle L -ordered fuzzifying convergence spaces and show that the new category and that of L -fuzzifying neighborhood spaces are isomorphic. Furthermore, each fibre on a fixed set of the category of L -fuzzifying neighborhood spaces and that of the category of principal L -ordered fuzzifying convergence spaces are isomorphic. At the end of the section, we propose that the category of principle L -ordered fuzzifying convergence spaces is a reflective subcategory of L -OFYC and it is a topological category. Again, this section is mostly motivated by reference [8].

Proposition 5.1. Let $(X, \lim) \in L$ -OFYC. The structure $\{N_{\lim}^x : 2^X \rightarrow L\}_{x \in X}$ defined by: For $x \in X$, $\forall U \in 2^X$, $N_{\lim}^x(U) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim \mathcal{F}(x) \rightarrow \mathcal{F}(U))$ is an L -fuzzifying neighborhood structure. We call it the induced L -fuzzifying neighborhood structure of (X, \lim) .

Theorem 3.5 suggests for $(X, \lim) \in L$ -OFYC the following definition.

Definition 5.2. Let \lim be an L -ordered fuzzifying convergence structure. If in addition the following condition (LYP) holds,

$$(LYP) \forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim \mathcal{F}(x) = S_F(N_{\lim}^x, \mathcal{F}),$$

then \lim is called a principal L -ordered fuzzifying convergence structure, and the pair (X, \lim) is called a principle L -ordered fuzzifying convergence space.

The full subcategory of L -OFYC consisting of all principle L -ordered fuzzifying convergence spaces is denoted by L -POFYC.

If an L -ordered fuzzifying convergence spaces satisfies (LYP), then a nice characterization of principle L -ordered convergence spaces in terms of L -fuzzifying neighborhood spaces is possible. We first need three theorems for preparation.

Theorem 5.3. *Let (X, N) be an L -fuzzifying neighborhood space. Then there exists a principle L -ordered fuzzifying convergence structure \lim on X satisfying $\forall x \in X, N_{\lim}^x = N^x$.*

Proof. For the L -fuzzifying neighborhood space (X, N) , define $\lim_N : \mathcal{F}_L(X) \rightarrow L^X$

$$\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim_N \mathcal{F}(x) = \bigwedge_{A \in 2^X} (N^x(A) \rightarrow \mathcal{F}(A)) = S_F(N^x, \mathcal{F}).$$

It is then readily checked that for (X, \lim_N) , the axiom (LY1), (OLY2), (LYP) hold. The properties of the residual implication of Theorem 2.1 are used.

$$(LY1): \forall x \in X, \lim_N x = \bigwedge_{A \in 2^X} (N^x(A) \rightarrow [x](A)) = 1.$$

(OLY2): In fact,

$$\begin{aligned} S(\lim_N \mathcal{F}, \lim_N \mathcal{G}) &= \bigwedge_{x \in X} (S_F(N^x, \mathcal{F}) \rightarrow S_F(N^x, \mathcal{G})) \\ &= \bigwedge_{x \in X} \left(\bigwedge_{A \in 2^X} (N^x(A) \rightarrow \mathcal{F}(A)) \rightarrow \bigwedge_{B \in 2^X} (N^x(B) \rightarrow \mathcal{G}(B)) \right) \\ &= \bigwedge_{x \in X} \bigwedge_{B \in 2^X} \left(\bigwedge_{A \in 2^X} (N^x(A) \rightarrow \mathcal{F}(A)) \rightarrow (N^x(B) \rightarrow \mathcal{G}(B)) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^X} ((N^x(B) \rightarrow \mathcal{F}(B)) \rightarrow (N^x(B) \rightarrow \mathcal{G}(B))) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^X} ((\mathcal{F}(B) \rightarrow \mathcal{G}(B))) \\ &= S_F(\mathcal{F}, \mathcal{G}). \end{aligned}$$

(LYP): For all $\mathcal{F} \in \mathcal{F}_L(X)$, we prove $\lim_N \mathcal{F}(x) = S_F(N_{\lim_N}^x, \mathcal{F})$. By the definition of \lim_N , $\lim_N \mathcal{F}(x) = S_F(N^x, \mathcal{F})$. It remains to verify that $N_{\lim_N}^x = N^x$. On one hand, for all $A \in 2^X$,

$$\begin{aligned} N_{\lim_N}^x(A) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_N \mathcal{F}(x) \rightarrow \mathcal{F}(A)) \\ &\leq \lim_N N^x(x) \rightarrow N^x(A) \\ &= N^x(A). \end{aligned}$$

On the other hand,

$$\begin{aligned} N_{\lim_N}^x(A) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_N \mathcal{F}(x) \rightarrow \mathcal{F}(A)) \\ &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left(\bigwedge_{B \in 2^X} (N^x(B) \rightarrow \mathcal{F}(B)) \rightarrow \mathcal{F}(A) \right) \\ &\geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (N^x(A) \rightarrow \mathcal{F}(A) \rightarrow \mathcal{F}(A)) \\ &\geq N^x(A). \end{aligned}$$

From this, the result follows by a standard argument. \square

In view of the above theorem, if N is an L -fuzzifying neighborhood structure, then there exists a principle L -ordered fuzzifying convergence structure \lim_N on X . Moreover, N_{\lim_N} is also an L -fuzzifying neighborhood structure and $N_{\lim_N} = N$. Conversely, we have the following theorem.

Theorem 5.4. *If \lim is a principle L -ordered fuzzifying convergence structure on X , then $\lim_{N_{\lim}} = \lim$.*

Proof. For all $\mathcal{F} \in \mathcal{F}_L(X), x \in X$, by (LYP), we have,
 $\lim_{N_{\lim}} \mathcal{F}(x) = \bigwedge_{A \in 2^X} (N_{\lim}^x(A) \rightarrow \mathcal{F}(A)) = S_F(N_{\lim}^x, \mathcal{F}) = \lim \mathcal{F}(x).$ \square

With respect to Theorem 5.3 and Theorem 5.4, we have a one-one correspondence between the objects of L -NGH and L -POFYC. The following theorem is about the relation between morphisms of them.

Theorem 5.5. *Let $(X, \lim^X), (Y, \lim^Y)$ be principle L -ordered fuzzifying convergence spaces, $(X, N_1), (Y, N_2)$ be L -fuzzifying neighborhood spaces, then we have*

(1) *If $f : (X, \lim^X) \rightarrow (Y, \lim^Y)$ is continuous, then $f : (X, N_{\lim^X}) \rightarrow (Y, N_{\lim^Y})$ is also continuous;*

(2) *If $f : (X, N_1) \rightarrow (Y, N_2)$ is continuous, then $f : (X, \lim_{N_1}^X) \rightarrow (Y, \lim_{N_2}^Y)$ is also continuous.*

Proof. (1) By the fact that $f : (X, \lim^X) \rightarrow (Y, \lim^Y)$ is continuous, we have $\forall x \in X, U \in 2^Y$,

$$\begin{aligned} N_{\lim^X}^x(f^{\leftarrow}(U)) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim^X \mathcal{F}(x) \rightarrow \mathcal{F}(f^{\leftarrow}(U))) \\ &\geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim^Y f^{\Rightarrow}(\mathcal{F})(f(x)) \rightarrow f^{\Rightarrow}(\mathcal{F})(U)) \\ &\geq \bigwedge_{\mathcal{G} \in \mathcal{F}_L(Y)} (\lim^Y \mathcal{G}(f(x)) \rightarrow \mathcal{G}(U)) \\ &= N_{\lim^Y}^{f(x)}(U), \end{aligned}$$

as desired.

(2) Conversely, by the fact that $f : (X, N_1) \rightarrow (Y, N_2)$ is continuous, we have $\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X$,

$$\begin{aligned} \lim_{N_2}^Y f^{\Rightarrow}(\mathcal{F})(f(x)) &= \bigwedge_{B \in 2^Y} (N_2^{f(x)}(B) \rightarrow f^{\Rightarrow}(\mathcal{F})(B)) \\ &\geq \bigwedge_{B \in 2^Y} (N_1^x(f^{\leftarrow}(B)) \rightarrow \mathcal{F}(f^{\leftarrow}(B))) \\ &\geq \bigwedge_{A \in 2^X} (N_1^x(A) \rightarrow \mathcal{F}(A)) \\ &= \lim_{N_1}^X \mathcal{F}(x), \end{aligned}$$

as desired. \square

By Theorems 5.3, 5.4 and 5.5, we actually have proved the following comprehensive theorem.

Theorem 5.6. *L -NGH is isomorphic to L -POFYC.*

Let X be a set. A fibre on X of the category of L -fuzzifying neighborhood spaces is denoted by $\mathbf{PrFN}_L(\mathbf{X})$. An order “ \leq ” on $\mathbf{PrFN}_L(\mathbf{X})$ can be defined by

$$N^1 \leq N^2 \Leftrightarrow \forall x \in X, A \in 2^X, N_x^1(A) \leq N_x^2(A).$$

A fibre on X of the category of principle L -ordered fuzzifying convergence spaces is denoted by $\mathbf{PFYC}_L(\mathbf{X})$. An order “ \leq ” on $\mathbf{PFYC}_L(\mathbf{X})$ can be defined as follows:

$$\lim_1 \leq \lim_2 \Leftrightarrow \forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x).$$

Then we have the following result.

Theorem 5.7. *$(\mathbf{PFYC}_L(\mathbf{X}), \leq)$ and $(\mathbf{PrFN}_L(\mathbf{X}), \leq)$ are isomorphic.*

Proof. Define a mapping: $h : \mathbf{PFYC}_L(\mathbf{X}) \rightarrow \mathbf{PrFN}_L(\mathbf{X})$, $\forall \lim \in \mathbf{PFYC}_L(\mathbf{X})$, $h(\lim) = N_{\lim}$, and a mapping: $k : \mathbf{PrFN}_L(\mathbf{X}) \rightarrow \mathbf{PFYC}_L(\mathbf{X})$, $\forall N \in \mathbf{PrFN}_L(\mathbf{X})$, $k(N) = \lim_N$. It has been verified in Theorems 5.3 and 5.4 that $h \circ k = id_{\mathbf{PrFN}_L(\mathbf{X})}$, $k \circ h = id_{\mathbf{PFYC}_L(\mathbf{X})}$. So h and k are both bijective. Furthermore, $k = h^{-1}$.

(1) For all $\lim_1, \lim_2 \in \mathbf{PFYC}_L(\mathbf{X})$, if $\lim_1 \leq \lim_2$, then for all $x \in X, A \in 2^X$,

$$\begin{aligned} N_{\lim_1}^x(A) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left(\lim_1 \mathcal{F}(x) \rightarrow \mathcal{F}(A) \right) \\ &\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left(\lim_2 \mathcal{F}(x) \rightarrow \mathcal{F}(A) \right) \\ &= N_{\lim_2}^x(A). \end{aligned}$$

So, $N_{\lim_1} \leq N_{\lim_2}$. i.e. $h(\lim_1) \leq h(\lim_2)$. Therefore, h is an order preserving map.

(2) For all $N_1, N_2 \in \mathbf{PrFN}_L(\mathbf{X})$, if $N_1 \leq N_2$, then for all $\mathcal{F} \in \mathcal{F}_L(X), x \in X$,

$$\begin{aligned} \lim_{N_1} \mathcal{F}(x) &= \bigwedge_{A \in 2^X} \left(N_1^x(A) \rightarrow \mathcal{F}(A) \right) \\ &\geq \bigwedge_{A \in 2^X} \left(N_2^x(A) \rightarrow \mathcal{F}(A) \right) \\ &= \lim_{N_2} \mathcal{F}(x). \end{aligned}$$

Hence, $\lim_{N_1} \leq \lim_{N_2}$, i.e. $h^{-1}(N_1) \leq h^{-1}(N_2)$. So h^{-1} is also an order preserving mapping.

From the above proof, we conclude that $(\mathbf{PFYC}_L(\mathbf{X}), \leq)$ and $(\mathbf{PrFN}_L(\mathbf{X}), \leq)$ are isomorphic. \square

At the end of this section, we propose the following results.

Theorem 5.8. *The category L -POFYC is a reflective subcategory of L -OFYC.*

Proof. Let $(X, \overline{\text{lim}}) \in L$ -OFYC and $E_{\overline{\text{lim}}} = \{ \text{lim} \mid (X, \text{lim}) \in L$ -POFYC, $\text{lim} \leq \overline{\text{lim}} \}$. Note that $E_{\overline{\text{lim}}}$ is not empty because it always contains lim_{sm} . Then we can construct a principal L -ordered fuzzifying convergence structure $\overline{\text{lim}}_* : \mathcal{F}_L(X) \rightarrow L^X$ as follows: For all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\overline{\text{lim}}_* \mathcal{F}(x) = \bigwedge_{\text{lim} \in E_{\overline{\text{lim}}}} \text{lim} \mathcal{F}(x)$. From this, we have

- (1) $id_X : (X, \overline{\text{lim}}) \rightarrow (X, \overline{\text{lim}}_*)$ is trivially continuous;
- (2) For a principal L -ordered fuzzifying convergence space (Y, lim^Y) , if $f : (X, \overline{\text{lim}}) \rightarrow (Y, \text{lim}^Y)$ is a continuous mapping, then $f : (X, \overline{\text{lim}}_*) \rightarrow (Y, \text{lim}^Y)$ is also continuous.

From the above facts, we immediately obtain that L -POFYC is a full reflective subcategory in L -OFYC. \square

Corollary 5.9. *The category L -POFYC is topological.*

6. The Relations Between L -fuzzifying Topological Spaces and Topological L -ordered Fuzzifying Convergence Spaces

In this section, we define another important subcategory of L -OFYC: the category of topological L -ordered fuzzifying convergence spaces. We will find out that the category mentioned above is isomorphic to L -FYS and to L -TNGH in case of a completely distributive lattice L . Furthermore, each fibre on X of the category of topological L -ordered fuzzifying convergence spaces is isomorphic to that of L -fuzzifying topological spaces and that of topological L -fuzzifying neighborhood spaces.

Definition 6.1. Let $(X, \text{lim}) \in L$ -POFYC, if in addition the mapping $\text{lim} : \mathcal{F}_L(X) \rightarrow L^X$ satisfies the following axiom:

$$(LYT) \forall U \in 2^X, N_{\text{lim}}^x(U) \leq \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N_{\text{lim}}^y(V),$$

then lim is called a topological L -ordered fuzzifying convergence structure, and (X, lim) is a topological L -ordered fuzzifying convergence space. The full subcategory of L -OFYC consisting of all topological L -ordered fuzzifying convergence spaces is denoted by L -TOFYC.

If lim is a topological L -ordered fuzzifying convergence structure, then a nice characterization of L -fuzzifying topologies is possible. We need two lemmas for preparation.

Lemma 6.2. *Let (X, N) be a topological L -fuzzifying neighborhood space, then (X, lim_N) is a topological L -ordered fuzzifying convergence space.*

Proof. As for (X, N) is a topological L -fuzzifying neighborhood space, (X, N) is an L -fuzzifying neighborhood space. By Theorem 5.3, we see $N = N_{\text{lim}_N}$. For N is a topological L -fuzzifying neighborhood structure, we know for all $x \in$

$X, \forall U \in 2^X, N^x(U) \leq \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N^y(V)$. Therefore, N_{\lim_N} satisfies (LYT). With Definition 6.1, the lemma holds. \square

Lemma 6.3. *Let $(X, \lim) \in L\text{-TOFYC}$, then there exists an L -fuzzifying topology τ on X such that $\lim_\tau = \lim$.*

Proof. Firstly, by Definition 6.1, $(X, \lim) \in L\text{-TOFYC}$ implies that N_{\lim} satisfies (N1) – (N4).

Secondly, let $\tau : 2^X \rightarrow L, \forall A \in 2^X, \tau(A) = \bigwedge_{x \in A} N_{\lim}^x(A)$. It can be easily proved that τ is an L -fuzzifying topology on X . Moreover, $N_\tau = N_{\lim}$. In fact, for all $x \in X, A \in 2^X$, we have by the axiom (LYT),

$$\begin{aligned} N_\tau^x(A) &= \bigvee_{x \in B \subseteq A} \tau(B) \\ &= \bigvee_{x \in B \subseteq A} \bigwedge_{y \in B} N_{\lim}^y(B) \\ &= N_{\lim}^x(A). \quad (\text{LYT}) \end{aligned}$$

With this and (LYP), we obtain for all $\mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim_\tau \mathcal{F}(x) = S_F(N_\tau^x, \mathcal{F}) = S_F(N_{\lim}^x, \mathcal{F}) = \lim \mathcal{F}(x)$.

Therefore, $\lim_\tau = \lim$ holds. \square

Lemmas 6.2, 6.3 together with the relations between topological L -fuzzifying neighborhood spaces and L -fuzzifying topological spaces in case that L is a completely distributive lattice show the following result.

Theorem 6.4. *If L is a completely distributive lattice, then $L\text{-FYS}, L\text{-TOFYC}$ and $L\text{-TNGH}$ are isomorphic to each other.*

We denote a fibre on X of the category of L -fuzzifying topological spaces by $\mathbf{FY}_L(\mathbf{X})$, and an order “ \leq ” on it is defined as follows: $\forall \tau_1, \tau_2 \in \mathbf{FY}_L(\mathbf{X})$,

$$\tau_1 \leq \tau_2 \Leftrightarrow \forall A \in 2^X, \tau_1(A) \leq \tau_2(A).$$

Denote a fibre on X of the category of topological L -fuzzifying neighborhood spaces by $\mathbf{FN}_L(\mathbf{X})$ and a fibre on X of the category of topological L -ordered fuzzifying convergence spaces by $\mathbf{TFYC}_L(\mathbf{X})$. In the same way as in the proof of Theorem 5.7, we obtain the following theorem trivially, and leave the straightforward proof for the interested reader.

Theorem 6.5. *$(\mathbf{FN}_L(\mathbf{X}), \leq), (\mathbf{FY}_L(\mathbf{X}), \leq), (\mathbf{TFYC}_L(\mathbf{X}), \leq)$ are isomorphic.*

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