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FIXED POINT THEOREM ON INTUITIONISTIC FUZZY METRIC SPACES

M. RAFI AND M. S. M. NOORANI

ABSTRACT. In this paper, we introduce intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces.

1. Introduction

The notion of intuitionistic fuzzy metric spaces was introduced and studied by Park in [5]. Saadati and Park in [6], further developed the theory of intuitionistic fuzzy topology (both in metric and normed) spaces. In this paper, we introduce an intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces. For the basic notions and concepts, we refer to [1, 3, 4, 5, 6].

2. Preliminaries

We review some basic concepts in intuitionistic fuzzy metric spaces as well as the intuitionistic fuzzy topology due to Saadati and Park [6].

Definition 2.1. [5, 6] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for all $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of continuous t-norm are

$$a * b = ab,$$

$$a * b = \min(a, b).$$

Definition 2.2. [5, 6] A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if it satisfies the following conditions: (i) \diamond is associative and commutative; (ii) \diamond is continuous; (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of t-conorm are

$$a \diamond b = \min(a + b, 1),$$

$$a \diamond b = \max(a, b).$$

Received: July 2005; Accepted: November 2005

Key words and Phrases: Intuitionistic fuzzy metric spaces, Fuzzy metric spaces, Fixed point theorem.

This work was supported by IRPA Grant No.: 09-02-02-0092-EA236.

Lemma 2.3. [5, 6] *If $*$ is a continuous t-norm and \diamond is continuous t-conorm, then:*

(i) *For every $a, b \in [0, 1]$, if $a > b$, there are $c, d \in [0, 1]$ such that $a * c \geq b$ and $a \geq b \diamond d$.*

(ii) *If $a \in [0, 1]$, there are $b, c \in [0, 1]$ such that $b * b \geq a$ and $a \geq c \diamond c$.*

The following definition is obtained from Mihet in [4].

Definition 2.4. [4] A fuzzy metric space in the sense of Kramosil and Michalek is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M: X^2 \times [0, \infty) \rightarrow [0, 1]$ is a mapping which satisfies the following properties for every $x, y, z \in X$:

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1, \forall t > 0 \Leftrightarrow x = y$;

(FM-3) $M(x, y, t) = M(y, x, t), \forall t > 0$;

(FM-4) $M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous;

(FM-5) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$;

(FM-6) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \forall t, s > 0$.

In $(X, M, *)$, the open ball $B_x(r, t)$ for $t > 0$ with center $x \in X$ and radius $r \in (0, 1)$ is defined as

$$(1) \quad B_x(r, t) = \{y \in X \mid M(x, y, t) > 1 - r\}.$$

The family $\{B_x(r, t) \mid x \in X, r \in (0, 1), t > 0\}$ is a neighborhood system for a Hausdorff topology on X induced by the fuzzy metric M . In a similar fashion, the dual space of $(X, M, *)$ is the fuzzy metric space (X, N, \diamond) defined below:

Definition 2.5. (New) A fuzzy metric space (X, N, \diamond) , where X is a nonempty set, \diamond is a continuous t-conorm and $N: X^2 \times [0, \infty) \rightarrow [0, 1]$ is a mapping assumed to satisfies the following properties for all $x, y, z \in X$:

(FM-D1) $N(x, y, 0) = 1$;

(FM-D2) $N(x, y, t) = 0, \forall t > 0 \Leftrightarrow x = y$;

(FM-D3) $N(x, y, t) = N(y, x, t), \forall t > 0$;

(FM-D4) $N(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous;

(FM-D5) $\lim_{t \rightarrow \infty} N(x, y, t) = 0$;

(FM-D6) $N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s), \forall t, s > 0$.

In (X, N, \diamond) , the open ball $D_x(r, t)$ for $t > 0$ with center $x \in X$ and radius $r \in (0, 1)$ is defined as

$$(2) \quad D_x(r, t) = \{y \in X \mid N(x, y, t) < r\}.$$

The family $\{D_x(r, t) \mid x \in X, r \in (0, 1), t > 0\}$ is a neighborhood's system for a Hausdorff topology on X induced by the fuzzy metric N . The following definition is introduced and studied by Park in [5].

Definition 2.6. [5] A 5-tuple $(X, M, N, *, \diamond)$ is called a intuitionistic fuzzy metric space if X is an arbitrary nonempty set, $*$ a continuous t-norm, \diamond a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(a) $M(x, y, t) + N(x, y, t) \leq 1$;

- (b) $M(x, y, t) > 0$;
- (c) $M(x, y, t) = 1 \Leftrightarrow x = y$;
- (d) $M(x, y, t) = M(y, x, t)$;
- (e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (f) $M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (g) $N(x, y, t) = N(y, x, t)$;
- (h) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;
- (i) $N(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is continuous..

The pair (M, N) is called an intuitionistic fuzzy metric on X . Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm $*$ and t-conorm \diamond are associated [6], i.e $x \diamond y = 1 - [(1 - x) * (1 - y)]$ for any $x, y \in X$.

Let $(X, M, N, *, \diamond)$ be a intuitionistic fuzzy metric space. For $t > 0$, the open ball $G_x(r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined by

$$(3) \quad G_x(r, t) = \{y \in X \mid M(x, y, t) > 1 - r, N(x, y, t) < r\}.$$

Note that it can be easily seen that $G_x = B_x \cap D_x$ where B_x and D_x as given by (1) and (2) respectively.

Since $*$ and \diamond are respectively a continuous t-norm and t-conorm, the family $\{G_x(r, t) \mid x \in X, r \in (0, 1), t > 0\}$ generates a topology $T_{(M, N)}$, called the (M, N) -topology (see [3, 6]). We have:

$A \in T_{(M, N)}$ if and only if $\forall x \in A, \exists t > 0, \exists r \in (0, 1)$ such that $G_x(r, t) \subset A$.

We denote the (M, N) -uniformity (or the uniformity generated by M , and N) by $U_{(M, N)}$. The family $\{U_{r, t}\}_{r \in (0, 1), t > 0}$, where

$$U_{r, t} = \{(x, y) \in X^2 \mid M(x, y, t) > 1 - r, N(x, y, t) < r\},$$

is a base for this uniformity.

Definition 2.7. [5, 6] Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space endowed with (m, n) -topology and $\{x_n\}$ in X . Then

- (i) $x_n \rightarrow x \Leftrightarrow M(x_n, x, t) \rightarrow 1$ and $N(x_n, x, t) \rightarrow 0$ as $n \rightarrow \infty$, for each $t > 0$.
- (ii) $\{x_n\}$ is called a (M, N) -Cauchy sequence if for each $r \in (0, 1)$ and $t > 0$, there exists an integer n_0 such that $M(x_n, x_m, t) > 1 - r$ and $N(x_n, x_m, t) < r$ for each $n, m \geq n_0$.
- (iii) The intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be (M, N) -complete if every (M, N) -Cauchy sequence is convergent.

3. Main Results

In the following sequel the letters \mathbb{N} and \mathbb{R}^+ denote the sets of positive integer numbers and positive real numbers, respectively.

Definition 3.1. [2] A quasi-metric on a set X is a function $d: X^2 \rightarrow \mathbb{R}^+$ satisfying the following conditions for every $x, y, z \in X$:

- (QM-1) $d(x, y) = 0$;
- (QM-2) $d(x, y) = d(y, x)$;
- (QM-3) $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 3.2. *Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space. For any $r \in (0, 1]$, we define $d: X^2 \rightarrow \mathbb{R}^+$ as follows:*

$$(4) \quad d_r(x, y) = \inf\{t > 0 \mid M(x, y, t) > 1 - r, N(x, y, t) < r\}$$

Then,

- (1) $(X, d_r : r \in (0, 1])$ is a generating space of a quasi-metric family.
- (2) the topology $T_{(d_r)}$ on $(X, d_r : r \in (0, 1])$ coincides with the (M, N) -topology on $(X, M, N, *, \diamond)$, (i.e., d_r is a compatible symmetric for $T_{(M, N)}$).

Proof. (1) From the definition of $\{d_r : r \in (0, 1]\}$, it is easy to see that $\{d_r : r \in (0, 1]\}$ satisfies the condition (QM-1) and (QM-2) of Definiton 3.1. Now we prove that $\{d_r : r \in (0, 1]\}$ also satisfies the condition (QM-3). Since $*$ and \diamond are continuous, by Lemma 2.3.(ii), for any given $r \in (0, 1)$, there exists $r' \in (0, r)$ such that

$$(1 - r') * (1 - r') > 1 - r$$

and

$$r' \diamond r' < r$$

. Setting $d_r(x, y) = a$ and $d_r(y, z) = b$, in equation (4), it follows that for any given $t > 0$,

$$M(x, y, a + t) > 1 - r', N(x, y, a + t) < r'$$

and

$$M(x, z, b + t) > 1 - r', N(y, z, b + t) < r'.$$

Whence

$$M(x, z, a + b + 2t) \geq M(x, y, a + t) * M(y, z, b + t) > (1 - r') * (1 - r') > 1 - r$$

and

$$N(x, z, a + b + 2t) \leq N(x, y, a + t) \diamond N(y, z, b + t) < r' \diamond r' < r.$$

Hence, we have

$$d_r(x, z) \leq a + b + 2t = d_r(x, y) + d_r(y, z) + 2t.$$

By the arbitrariness of $t > 0$, we have

$$d_r(x, z) \leq d_r(x, y) + d_r(y, z).$$

- (2) To prove this condition, it is only necessary to show that for any $t > 0$ and $r \in (0, 1)$

$$d_r(x, y) < t \Leftrightarrow M(x, y, t) > 1 - r, N(x, y, t) < r.$$

In fact, if $d_r(x, y) < t$, then by (4), we have $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$. Conversely, if $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$, since M and N are continuous functions, there exists an $s > 0$ such that $M(x, y, t - s) > 1 - r$ and $N(x, y, t - s) < r$ and so $d_r(x, y) \leq t - s < t$. This completes the proof. \square

In fuzzy metric spaces $(X, M, *)$, the map $f: X \rightarrow X$ is said to be a fuzzy contraction if there exists $k \in (0, 1)$ such that

$$(5) \quad \frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t) - 1} \right), \forall x, y \in X, \forall t > 0.$$

Several fixed point theorem has been proved by using (5) in a fuzzy metric spaces (see [1, 7]).

If the fuzzy metric space (X, N, \diamond) is a dual space of $(X, M, *)$, the map $f: X \rightarrow X$ in (X, N, \diamond) may also enjoy a fuzzy contractive condition. Since the definition of the dual space (X, N, \diamond) is similar in the sense of metric space, we can consider the following new contractive condition for a self mapping f in (X, N, \diamond) .

Definition 3.3. (New) Let (X, N, \diamond) be a fuzzy metric space. The map $f: X \rightarrow X$ is a fuzzy contraction in (X, N, \diamond) if there exists $k \in (0, 1)$ such that

$$(6) \quad N(f(x), f(y), t) \leq kN(x, y, t), \forall x, y \in X, \forall t > 0.$$

By using the contractive conditions (5) and (6), now we are able to define an intuitionistic fuzzy contractive map f as follow:

Definition 3.4. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. We say that the mapping $f: X \rightarrow X$ is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t) - 1} \right)$$

and

$$N(f(x), f(y), t) \leq kN(x, y, t),$$

for each $x, y \in X$ and $t > 0$.

We give a definition for a intuitionistic fixed point theorem in intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$.

Definition 3.5. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and let $f: X \rightarrow X$ be an intuitionistic fuzzy contractive mapping. Then there exists $z \in X$ such that $z = f(z)$ (We call z an intuitionistic fuzzy fixed point of f).

Now, we prove the following theorem.

Theorem 3.6. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space. Let $f: X \rightarrow X$ be an intuitionistic fuzzy contractive mapping. Then f has a unique intuitionistic fixed point.*

Proof. We fix $x_0 \in X$. Let $x_{n+1} = f(x_n), n \in \mathbb{N}$. We have for $n > m$ and $t > 0$,

$$\begin{aligned} \frac{1}{M(x_n, x_{n+m}, t)} - 1 &= \frac{1}{M(f(x_{n-1}), f(x_{n+m-1}), t)} - 1 \\ &\leq k \left(\frac{1}{M(x_{n-1}, x_{n+m-1}, t)} - 1 \right) \\ &= k \left(\frac{1}{M(f(x_{n-2}), f(x_{n+m-2}), t)} - 1 \right) \\ &\vdots \\ &\leq k^n \left(\frac{1}{M(x_0, x_m, t)} - 1 \right). \end{aligned}$$

Whence, for $n > m, \frac{1}{M(x_n, x_{n+m}, t)} - 1 \rightarrow 0$ as $n \rightarrow \infty$, that is $M(x_n, x_{n+m}, t) \rightarrow 1$ as $n \rightarrow \infty$. Also, for $n > m$ and $t > 0$, by (6) we have

$$\begin{aligned} N(x_n, x_{n+m}, t) &= N(f(x_{n-1}), f(x_{n+m-1}), t) \\ &\leq kN(x_{n-1}, x_{n+m-1}, t) \\ &= kN(f(x_{n-2}), f(x_{n+m-2}), t) \\ &\vdots \\ &\leq k^n N(x_0, x_m, t). \end{aligned}$$

Whence, for $n > m, N(x_n, x_{n+m}, t) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we can conclude that $\{x_n\}$ is a Cauchy sequence in X . Since $(X, M, N, *, \diamond)$ is complete, the sequence $\{x_n\}$ converges to some $y \in X$. We show that y is an intuitionistic fixed point of f , i.e $y = f(y)$. By the contractive condition (5) of f , we have

$$\frac{1}{M(f(y), f(x_n), t)} - 1 \leq k \left(\frac{1}{M(y, x_n, t)} - 1 \right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence, $\lim_{n \rightarrow \infty} M(f(y), f(x_n), t) = 1$ for every $t > 0$. And by (6), we have:

$$N(f(y), f(x_n), t) \leq kN(y, x_n, t) \rightarrow 0 \text{ as } n \rightarrow \infty$$

. Thus, $\lim_{n \rightarrow \infty} N(f(y), f(x_n), t) = 0$ for every $t > 0$. In both cases, we have $\lim_{n \rightarrow \infty} f(x_n) = f(y)$, i.e., $\lim_{n \rightarrow \infty} x_{n+1} = f(y)$, therefore $y = f(y)$. For uniqueness, assume $z = f(z)$ for some $z \in X$. Then for $t > 0$, we have

$$\begin{aligned} \frac{1}{M(y, z, t)} - 1 &= \frac{1}{M(f(y), f(z), t)} - 1 \\ &\leq k \left(\frac{1}{M(y, z, t)} - 1 \right) \\ &\vdots \\ &\leq k^n \left(\frac{1}{M(y, z, t)} - 1 \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

whence, for every $t > 0$ we have $M(y, z, t) = 1$, it follows that $z = y$. This completes the proof. \square

Acknowledgements. The authors would like to express their sincere thanks to the referees for their useful suggestions and comments as well as some aspects about notion when reading this paper.

REFERENCES

- [1] V. Gregory and A. Sapena, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, **125** (2002), 245-253.
- [2] H. P. A. Kunzi and S. Romaguera, *Quasi-metric spaces, quasi-metric hyperspaces and uniform local compactness*, Rend. Istit. Univ. Trieste, **30** (1999), 133-144.
- [3] R. Lowen, *Fuzzy set theory*, Kluwer Academic Publisher, Dordrecht, 1996.
- [4] D. Mihet, *A Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets and Systems, **144** (2004), 431-439.
- [5] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, **22** (2004), 1039-1046.
- [6] R. Saadati and J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons and Fractals, **27** (2006), 331-344.
- [7] P. Vijayaraju and M. Marudai, *Fixed point theorem for fuzzy mappings*, Fuzzy Sets and Systems, **135** (2003), 401-408.

MOHD. RAFI SEGI RAHMAT*, SCHOOL OF MATHEMATICAL SCIENCE, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITI KEBANGSAAN MALAYSIA, 43600 BANGI, SELANGOR D.E., MALAYSIA
E-mail address: mdrafzi@yahoo.com

MOHD. SALMI MD. NOORANI, SCHOOL OF MATHEMATICAL SCIENCE, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITI KEBANGSAAN MALAYSIA, 43600 BANGI, SELANGOR D.E., MALAYSIA
E-mail address: msn@pkrisc.cc.ukm.my

*CORRESPONDING AUTHOR

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