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کارگاه آنلاین آشنایی با پایگاه های اطلاعات علمی بین المللی و ترکیه های جستجو

POINTWISE PSEUDO-METRIC ON THE  $L$ -REAL LINE

F. -G. SHI

ABSTRACT. In this paper, a pointwise pseudo-metric function on the  $L$ -real line is constructed. It is proved that the topology induced by this pointwise pseudo-metric is the usual topology.

## 1. Introduction

The  $L$ -fuzzy unit interval and the  $L$ -fuzzy real line are two important  $L$ -topological spaces. The  $L$ -fuzzy unit interval was defined by Hutton [2]. The  $L$ -fuzzy real line was respectively defined by Höhle [3] and Gantner et al. [4]. They are important not only in  $L$ -topology, but also in other fields.

To reflect the characteristics of pointwise  $L$ -topology, i.e., the relation between a fuzzy point and its Q-neighborhoods (or R-neighborhoods) [5], a theory of pointwise uniformities and a theory of pointwise metrics were introduced on completely distributive lattices and in  $L$ -fuzzy set theory (see [6, 7, 8, 9]). Many ideal results in general topology were generalized to  $L$ -topology. In [9], it was proved that the  $L$ -fuzzy real line is pointwise pseudo-metrizable, but no pointwise pseudo-metric function on the  $L$ -fuzzy real line was given. In this paper, our aim is to construct a pointwise pseudo-metric function in the  $L$ -real line and prove that the topology induced by this pointwise pseudo-metric function is the usual topology.

## 2. Preliminaries

Throughout this paper,  $L$  always denotes a completely distributive lattice with an order-reversing involution.  $M(L^X)$  denotes the set of all non-zero  $\vee$ -irreducible elements in  $L^X$ . For  $A \in L^X$ ,  $\beta(A)$  denotes the maximal minimal family of  $A$  (see [5]) and  $\beta^*(A) = \beta(A) \cap M(L^X)$ . It is easy to verify that for  $e \in M(L)$ ,  $e \in \beta^*(A)$  if and only if  $a \ll A$ , where  $\ll$  is the way below relation ([1]).

**Definition 2.1** ([9]). A pointwise pseudo-quasi-metric on  $L^X$  is a mapping  $d : M(L^X) \times M(L^X) \rightarrow [0, +\infty)$  satisfying the following (M1)–(M3):

$$(M1) \quad \forall a \in M(L^X), \quad d(a, a) = 0.$$

$$(M2) \quad \forall a, b, c \in M(L^X), \quad d(a, c) \leq d(a, b) + d(b, c).$$

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$$(M3) \forall a, b \in M(L^X), d(a, b) = \bigwedge_{c \ll b} d(a, c).$$

A pointwise pseudo-quasi-metric  $d$  is called a pointwise pseudo-metric if it satisfies the following conditions.

$$(M4) \forall a, b, c \in M(L^X), a \leq b \text{ implies } d(a, c) \leq d(b, c).$$

$$(M5) \forall \lambda, \mu \in M(L^X), \bigwedge_{a \not\leq \lambda'} d(a, \mu) < r \text{ if and only if } \bigwedge_{b \not\leq \mu'} d(b, \lambda) < r.$$

**Theorem 2.2** ([9]). *Let  $d$  be a pointwise pseudo-metric on  $L^X$ .  $\forall r \in (0, +\infty)$ , define a mapping  $P_r : M(L^X) \rightarrow L^X$  by*

$$P_r(a) = \bigvee \{b \in M(L^X) \mid d(a, b) \geq r\}.$$

*Then the family  $\{P_r \mid r \in (0, +\infty)\}$  of  $R$ -nbd mappings of  $d$  satisfies the following conditions.*

$$(R1) \forall a \in M(L^X), \bigwedge_{r > 0} P_r(a) = 0;$$

$$(R2) \forall a \in M(L^X), \forall r \in (0, +\infty), a \not\leq P_r(a);$$

$$(R3) \forall r, s \in (0, +\infty), P_s \odot P_r \geq P_{r+s};$$

$$(R4) \forall a \in M, P_r(a) = \bigwedge_{s < r} P_s(a);$$

$$(R5) \forall r \in (0, +\infty), P_r \text{ is symmetric.}$$

**Theorem 2.3** ([9]). *If  $\{P_r \mid P_r : M(L^X) \rightarrow L^X, r \in (0, +\infty)\}$  is a family of mappings satisfying (R1)–(R5), and we define  $d : M(L^X) \times M(L^X) \rightarrow [0, +\infty)$  by*

$$d(a, b) = \bigwedge \{r \mid b \not\leq P_r(a)\},$$

*then  $d$  is a pointwise pseudo-metric on  $L^X$  and the family of  $R$ -nbd mappings of  $d$  is exactly  $\{P_r \mid r \in (0, +\infty)\}$ .*

**Theorem 2.4** ([9]). *If  $d$  is a pointwise pseudo-quasi-metric on  $L^X$ , then*

(1)  $\{P_r(a) \mid a \in M(L^X), r \in (0, +\infty)\}$  *is a base for a co-topology on  $L^X$ . This co-topology is denoted by  $\eta_d$ ;*

(2)  $\{P_r(a) \mid r > 0\}$  *is a locally  $R$ -neighborhood base at  $a$  in the co-topology  $\eta_d$ .*

**Definition 2.5** ([3, 4]). *The  $L$ -(fuzzy) real line  $\mathbb{R}(L)$  is defined as the set of all equivalence classes of antitone maps  $\lambda : \mathbb{R} \rightarrow L$  satisfying*

$$\bigvee_{t \in \mathbb{R}} \lambda(t) = 1 \text{ and } \bigwedge_{t \in \mathbb{R}} \lambda(t) = 0,$$

where the equivalence identifies two maps  $\lambda$  and  $\mu$  if and only if  $\forall t \in I, \lambda(t+) = \mu(t+)$ . The canonical  $L$ -topology on  $\mathbb{R}(L)$  is generated from the subbase  $\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}$ , where

$$\mathcal{L}_t : I(L) \rightarrow L \text{ by } \mathcal{L}_t(\lambda) = \lambda(t-)'$$

$$\mathcal{R}_t : I(L) \rightarrow L \text{ by } \mathcal{R}_t(\lambda) = \lambda(t+).$$

### 3. Pointwise Pseudo-metric on the $L$ -real Line

**Lemma 3.1.** *Let  $\mathbb{R}(L)$  be the  $L$ -real line. Define a mapping  $\varepsilon : M(L^{\mathbb{R}(L)}) \rightarrow \mathbb{R}$  and a mapping  $\sigma : M(L^{\mathbb{R}(L)}) \rightarrow \mathbb{R}$  such that for all  $e \in M(L^{\mathbb{R}(L)})$ ,*

$$\varepsilon(e) = \sup \{t \mid e \leq \mathcal{L}'_t\}, \quad \sigma(e) = \inf \{t \mid e \leq \mathcal{R}'_t\},$$

*Then we have the following results:*

- (1)  $\varepsilon(e) = \max\{t \mid e \leq \mathcal{L}'_t\}$ ,  $\sigma(e) = \min\{t \mid e \leq \mathcal{R}'_t\}$ .
- (2) *If  $a, b \in M(L^{\mathbb{R}(L)})$  and  $a \leq b$ , then  $\varepsilon(a) \geq \varepsilon(b)$  and  $\sigma(a) \leq \sigma(b)$ .*
- (3) *If  $b \in M(L^{\mathbb{R}(L)})$ , then  $\varepsilon(b) = \bigwedge_{c \ll b} \varepsilon(c)$  and  $\sigma(b) = \bigvee_{c \ll b} \sigma(c)$ .*
- (4)  $\forall \lambda, \mu \in M(L^{\mathbb{R}(L)})$ , *there exists a  $\not\leq \lambda'$  such that  $\varepsilon(\mu) < \varepsilon(a) + r$  if and only if there exists  $b \not\leq \mu'$  such that  $\sigma(\lambda) > \sigma(b) - r$ .*

*Proof.* (1) and (2) are obvious. By (2) we can obtain that  $\varepsilon(b) \leq \bigwedge_{c \ll b} \varepsilon(c)$  and  $\sigma(b) \geq \bigvee_{c \ll b} \sigma(c)$ . Thus in order to prove (3) we need only to prove that

$$\varepsilon(b) \geq \bigwedge_{c \ll b} \varepsilon(c) \quad \text{and} \quad \sigma(b) \leq \bigvee_{c \ll b} \sigma(c).$$

Suppose that  $\varepsilon(b) < \bigwedge_{c \ll b} \varepsilon(c)$ . Then there exists  $s \in \mathbb{R}$  such that

$$\varepsilon(b) = \max\{t \mid b \leq \mathcal{L}'_t\} < s < \bigwedge_{c \ll b} \varepsilon(c).$$

This implies that  $b \not\leq \mathcal{L}'_s$ . Further there exists  $c \ll b$  such that  $c \leq \mathcal{L}'_s$ . Thus we have that  $\varepsilon(c) < s$ . By  $s < \bigwedge_{c \ll b} \varepsilon(c)$  we obtain a contradiction. Therefore  $\varepsilon(b) \geq \bigwedge_{c \ll b} \varepsilon(c)$ . Similarly we can prove that  $\sigma(b) \leq \bigvee_{c \ll b} \sigma(c)$ . Hence (3) follows.

To prove (4) suppose that  $\varepsilon(\mu) < \varepsilon(a) + r$ . Then there is  $t > 0$  such that  $\varepsilon(\mu) < \varepsilon(a) + r - t$ . This implies that

$$\mu \not\leq \mathcal{L}'_{\varepsilon(a)+r-t} \quad \text{or} \quad \mathcal{L}_{\varepsilon(a)+r-t} \not\leq \mu'.$$

So there exists a point  $b \leq \mathcal{L}_{\varepsilon(a)+r-t}$  such that  $b \not\leq \mu'$ . We obtain

$$\sigma(b) \leq \varepsilon(a) + r - t \quad \text{or} \quad \sigma(b) - r < \varepsilon(a)$$

since  $\mathcal{L}_{\varepsilon(a)+r-t} \leq \mathcal{R}'_{\varepsilon(a)+r-t}$ . By  $a \leq \mathcal{L}'_{\varepsilon(a)}$  we have that

$$\lambda \not\leq a' \geq \mathcal{L}_{\varepsilon(a)} \geq \mathcal{R}'_{\sigma(b)-r}.$$

Therefore  $\sigma(\lambda) > \sigma(b) - r$ . □

**Theorem 3.2.** *Let  $\mathbb{R}(L)$  be the  $L$ -real line. For all  $a, b \in M(L^{\mathbb{R}(L)})$ , define*

$$d_1(a, b) = \max\{\varepsilon(b) - \varepsilon(a), 0\}, \quad d_2(a, b) = \max\{\sigma(a) - \sigma(b), 0\},$$

*Then  $d_1, d_2$  are pointwise pseudo-quasi-metrics,  $\{\mathcal{L}_t \mid t \in \mathbb{R}\}$  is the topology induced by  $d_1$  and  $\{\mathcal{R}_t \mid t \in \mathbb{R}\}$  is the topology induced by  $d_2$ .*

*Proof.* We only prove that  $d_1$  is a pointwise pseudo-quasi-metric. The proof for  $d_2$  is similar. Obviously, by (2) in Lemma 3.1 we know that  $a \leq b \Rightarrow d_1(a, b) = 0$ . Thus (M1) is true. (M2) can be obtained as follows.

$$\begin{aligned} d_1(a, c) &= \max\{\varepsilon(c) - \varepsilon(a), 0\} \\ &= \max\{\varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), 0\} \\ &\leq \max\{\varepsilon(b) - \varepsilon(a), 0\} + \max\{\varepsilon(c) - \varepsilon(b), 0\} \\ &= d_1(a, b) + d_1(b, c) \end{aligned}$$

(M3) can be obtained as follows:

$$\begin{aligned} d_1(a, b) &= \max\{\varepsilon(b) - \varepsilon(a), 0\} \\ &= \max\left\{\bigwedge_{c \ll b} \varepsilon(c) - \varepsilon(a), 0\right\} \\ &= \max\left\{\bigwedge_{c \ll b} (\varepsilon(c) - \varepsilon(a)), 0\right\} \\ &= \bigwedge_{c \ll b} \max\{\varepsilon(c) - \varepsilon(a), 0\} = \bigwedge_{c \ll b} d_1(a, c). \end{aligned}$$

In order to prove that  $\{\mathcal{L}_t \mid t \in \mathbb{R}\}$  is the topology induced by  $d_1$  and  $\{\mathcal{R}_t \mid t \in \mathbb{R}\}$  is the topology induced by  $d_2$ , we only need to prove that the family  $\{P_r^{d_1} \mid r > 0\}$  of R-nbd mappings of  $d_1$  and the family  $\{P_r^{d_2} \mid r > 0\}$  of R-nbd mappings of  $d_2$  satisfy the following condition:

$$P_r^{d_1}(a) = \mathcal{L}'_{\varepsilon(a)+r} \quad \text{and} \quad P_r^{d_2}(a) = \mathcal{R}'_{\sigma(a)-r}.$$

In fact,  $\forall a, b \in M(L^{\mathbb{R}(L)})$ , we have:

$$\begin{aligned} b \leq P_r^{d_1}(a) &\Leftrightarrow d_1(a, b) \geq r \\ &\Leftrightarrow \varepsilon(b) - \varepsilon(a) \geq r \\ &\Leftrightarrow \varepsilon(b) \geq \varepsilon(a) + r \Leftrightarrow b \leq \mathcal{L}'_{\varepsilon(a)+r} \end{aligned}$$

and

$$\begin{aligned} b \leq P_r^{d_2}(a) &\Leftrightarrow d_2(a, b) \geq r \\ &\Leftrightarrow \sigma(a) - \sigma(b) \geq r \\ &\Leftrightarrow \sigma(b) \leq \sigma(a) - r \Leftrightarrow b \leq \mathcal{R}'_{\sigma(a)-r} \end{aligned}$$

The result follows. □

**Remark 3.3.** When  $L = 2$ ,  $d_1$  and  $d_2$  are conjugate pseudo-quasi-metrics in the usual sense.

**Theorem 3.4.** Let  $\mathbb{R}(L)$  be the  $L$ -real line. For all  $a, b \in M(L^{\mathbb{R}(L)})$ , define

$$d(a, b) = \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} = \max\{d_1(a, b), d_2(a, b)\}.$$

Then  $d$  is a pointwise pseudo-metric and  $d$  exactly induces the topology on  $\mathbb{R}(L)$ .

**Proof.** By (2) in Lemma 3.1 it is obvious that we know that  $a \leq b \Rightarrow d(a, b) = 0$ . Thus (M1) is true. (M2) can be obtained as follows:

$$\begin{aligned} d(a, c) &= \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} \\ &= \max\{\varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b) + \sigma(b) - \sigma(c), 0\} \\ &\leq \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} + \max\{\varepsilon(c) - \varepsilon(b), \sigma(b) - \sigma(c), 0\} \\ &= d(a, b) + d(b, c) \end{aligned}$$

(M3) can be obtained as follows:

$$\begin{aligned} d(a, b) &= \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} \\ &= \max\left\{\bigwedge_{c \ll b} \varepsilon(c) - \varepsilon(a), \sigma(a) - \bigvee_{c \ll b} \sigma(c), 0\right\} && \text{by Lemma 3.1} \\ &= \max\left\{\bigwedge_{c \ll b} (\varepsilon(c) - \varepsilon(a)), \bigwedge_{c \ll b} (\sigma(a) - \sigma(c)), 0\right\} \\ &= \bigwedge_{c \ll b} \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} = \bigwedge_{c \ll b} d(a, c) \end{aligned}$$

(M4) can be obtained from (2) in Lemma 3.1.

To prove (M5), we note that  $\forall \lambda, \mu \in M(L^{\mathbb{R}(L)})$ , if

$$\bigwedge_{a \not\leq \lambda'} d(a, \mu) = \bigwedge_{a \not\leq \lambda'} \max\{\varepsilon(\mu) - \varepsilon(a), \sigma(a) - \sigma(\mu), 0\} < r,$$

then there exists  $a \not\leq \lambda'$  such that

$$\max\{\varepsilon(\mu) - \varepsilon(a), \sigma(a) - \sigma(\mu), 0\} < r,$$

i.e.,

$$\varepsilon(\mu) - \varepsilon(a) < r, \quad \sigma(a) - \sigma(\mu) < r.$$

Hence we have that

$$\varepsilon(\mu) < \varepsilon(a) + r, \quad \sigma(\mu) > \sigma(a) - r.$$

By (4) in Lemma 3.1 we know that there exist  $b \not\leq \mu'$  and  $c \not\leq \mu'$  such that

$$\sigma(\lambda) > \sigma(b) - r, \quad \varepsilon(\lambda) < \varepsilon(c) + r.$$

Thus, since  $\mu'$  is a prime element,  $b \wedge c \not\leq \mu'$ . Take a point  $d \leq b \wedge c$  such that  $d \not\leq \mu'$ . Then

$$\sigma(\lambda) > \sigma(b) - r \geq \sigma(d) - r, \quad \varepsilon(\lambda) < \varepsilon(c) + r \leq \varepsilon(d) + r.$$

This implies that

$$\bigwedge_{d \not\leq \mu'} d(d, \lambda) = \bigwedge_{d \not\leq \mu'} \max\{\varepsilon(\lambda) - \varepsilon(d), \sigma(d) - \sigma(\lambda), 0\} < r.$$

In order to prove that  $\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}$  is a subbase of the topology induced by  $d$ , we only need to prove that the family  $\{P_r^d \mid r > 0\}$  of R-nbd mappings of  $d$  satisfies the following condition:

$$P_r^d(a) = \mathcal{L}'_{\varepsilon(a)+r} \vee \mathcal{R}'_{\sigma(a)-r}.$$

In fact,  $\forall a, b \in M(L^{\mathbb{R}(L)})$  we have:

$$\begin{aligned} b \leq P_r^d(a) &\Leftrightarrow d(a, b) \geq r \\ &\Leftrightarrow \varepsilon(b) - \varepsilon(a) \geq r \text{ or } \sigma(a) - \sigma(b) \geq r \\ &\Leftrightarrow \varepsilon(b) \geq \varepsilon(a) + r \text{ or } \sigma(b) \leq \sigma(a) - r \\ &\Leftrightarrow b \leq \mathcal{L}'_{\varepsilon(a)+r} \vee \mathcal{R}'_{\sigma(a)-r} \end{aligned}$$

The result follows.  $\square$

**Remark 3.5.** When  $L = 2$ , the pointwise pseudo-metric  $d$  in Theorem 3.4 can be regarded as the usual pseudo-metric defined by  $d(a, b) = |a - b|$ .

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