CATEGORY OF \((POM)_L\)-FUZZY GRAPHS AND HYPERGRAPHS

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Abstract. In this note by considering a complete lattice \(L\), we define the notion of an \(L\)-Fuzzy hyperrelation on a given non-empty set \(X\). Then we define the concepts of \((POM)_L\)-Fuzzy graph, hypergraph and subhypergroup and obtain some related results. In particular we construct the categories of the above mentioned notions, and give a (full and faithful) functor form the category of \((POM)_L\)-Fuzzy subhypergroups ((\(POM\))\(_L\)-Fuzzy graphs) into the category of \((POM)_L\)-Fuzzy hypergraphs. Also we show that for each finite objects in the category of \((POM)_L\)-Fuzzy graphs, the coproduct exists, and under a suitable condition the product also exists.

1. Introduction

Rosenfeld [9] in 1975 defined the notion of a fuzzy graph. Berge studied hypergraphs [1]. Roy and Goetschel gave the notion of fuzzy hypergraphs [10]. Zahedi and Khorashadi-Zadeh in [9] gave some categoric connections between fuzzy hypergraphs, subhypergroups, . . . . Now we follow [10] and [11]. In this regard we redefine the notion of fuzzy hypergraph. In fact we give a new approach to this notion. To explain this, first we give the notions of fuzzy graph and fuzzy hypergraph which have defined in [9] and [10] respectively.

Definition 1.1. [9] A fuzzy graph is a triple \((X, \delta, \mu)\), where \(\delta\) is a fuzzy subset of a finite non-empty set of \(X\) and \(\mu\) is a fuzzy relation on \(\delta\), i.e. \(\mu\) is a fuzzy subset of \(X \times X\), and \(\mu(x, y) \leq \delta(x) \wedge \delta(y)\), for all \(x, y \in X\).

Definition 1.2. [10] Let \(X\) be a finite non-empty set and let \(\xi\) be a finite family of non-trivial fuzzy subsets on \(X\), i.e. for all \(\mu\) in \(\xi\), \(\operatorname{supp}\mu \neq \emptyset\) and \(X = \bigcup_{\mu \in \xi} \operatorname{supp}\mu\), where by \(\operatorname{supp}\mu\) we mean the set \(\{x \in X | \mu(x) > 0\}\). Then the pair \(\mathcal{H} = (X, \xi)\) is called a fuzzy hypergraph on \(X\).

Remark 1.3. We expect a hypergraph to be a fuzzy graph if \(\xi\) in Definition 1.2 is a singleton. However if \(\delta\) is a fuzzy subset on a finite nonempty set \(X\) and \(\operatorname{supp}\delta = X\), then the pair \((X, \xi = \{\delta\})\) is a fuzzy hypergraph on \(X\) according to Definition 1.2, while \((X, \delta)\) is not a fuzzy graph according to Definition 1.1. The problem arises because the fuzzy relation \(\mu\) has no place in Definition 1.2. Hence this definition is not a generalization of Definition 1.1. So in this paper we work with the Definition 1.2, that is a genuine extension of Definition 1.1.
Now in this note first we give the notions of $L$-Fuzzy hypersubsets, $L$-Fuzzy hyperrelations on a set $X$, $(POM)_{L}$-$Fuzzy hyperrelations on an $L$-Fuzzy hypersubsets of $X$ and $(POM)_{L}$-$Fuzzy hyperrelations on a finite family of $L$-Fuzzy subsets. Then we present the concepts of $(POM)_{L}$-$Fuzzy hyperrelations on a finite family of $L$-Fuzzy subsets. After that we show that there is a full and faithful functor from $(POM)_{L}$-$FG$ into $(POM)_{L}$-$FHG$. Finally we define the notion of $(POM)_{L}$-$Fuzzy hypergroups and then construct the category $(POM)_{L}$-$FHG$ of all $(POM)_{L}$-$Fuzzy hypergroups, and obtain some related results.

Throughout this paper we let $L$ be a complete lattice with the greatest element 1 and the least element 0.

**Definition 1.4.** [3, 8] Let $T : L \times L \to L$ be a binary operation having the properties:

(i) $T(x, 1) = x$
(ii) $T(x, y) = T(y, x)$
(iii) $T(x, y) \leq T(u, y)$ if $x \leq u$
(iv) $T(x, T(y, z)) = T(T(x, y), z)$.

Henceforth $(L, T)$ is a partially ordered commutative monoid [2].

**Remark 1.5.** In the mathematical tradition of algebra $(L, T)$ is better known as a partially ordered monoid in which the unity coincides with the top element of the lattice, for example in this regard see [2, 5]. However authors in the fuzzy set tradition sometimes call $T$ an $L$-t-norm, because $T$ is known as a t-norm when $L$ is the unit interval.

It is obvious that if $\{x_\alpha\}_{\alpha \in \Lambda}$ and $\{y_\beta\}_{\beta \in \Lambda'}$ are two families of elements of $L$, then

\[ \bigvee_{\alpha \in \Lambda} T(x_\alpha, y_\beta) \leq T\left(\bigvee_{\alpha \in \Lambda} x_\alpha, \bigvee_{\beta \in \Lambda'} y_\beta\right). \]  

By an $L$-Fuzzy subset $\delta$ on a set $X$, we mean a function $\delta : X \to L$.

**Notation:**

(i) We let $\mathcal{F}_L(X)$ shows the set of all $L$-Fuzzy subsets on $X$, i.e.

\[ \mathcal{F}_L(X) = \{\delta : X \to L \text{ is a function}\}. \]

(ii) Write $p^{*}(X) = p(X) \setminus \{\emptyset\}$, i.e. $p^{*}(X)$ is the set of all non-empty subsets of $X$.

**Definition 1.6.** [7] It is well-known that a hyperstructure is a non-empty set $H$ together with a map $o : H \times H \to p^{*}(H)$, called hyperoperation. A hyperstructure $(H, o)$ is called a hypergroup if the following axioms hold:

(i) $(xoy)o z = xo(yoz), \forall x, y, z \in H$,

(ii) $aoH = H = Hoa, \forall a \in H$,

where by $AoB$ we mean $\bigcup_{x \in A, y \in B} xoy$, for all subsets $A, B$ of $H$.

**Lemma 1.7.** Let $(H, o)$ be a hyperstructure. Then the following statements are equivalent:
(i) \( aoH = Hoa = H, \forall a \in H \),
(ii) for all \( a \) and \( y \) in \( H \) there exist \( u \) and \( v \) in \( H \) such that \( y \in uoa \) and \( y \in aov \).

2. \((POM)_L\)-Fuzzy (hyper)graphs

**Definition 2.1.** Let \( \delta \in F_L(X) \). Then we say that \( \delta \) is non-trivial if

\[ \delta^* = \text{supp} \delta = \{ x \in X | 0 \leq \delta(x), \delta(x) \neq 0 \} \neq \emptyset. \]

**Definition 2.2.** Let \( \mu \in F_L(X \times X) \). Then we say that \( \mu \) is an \( L \)-Fuzzy relation on \( X \).

**Definition 2.3.** Let \( \delta \in F_L(X) \) and \( \mu \in F_L(X \times X) \). Then \( \mu \) is said to be a \((POM)_L\)-Fuzzy relation on \( \delta \) if

\[ \mu(x, y) \leq T(\delta(x), \delta(y)), \forall x, y \in X. \]

**Definition 2.4.** Let \( X \neq \emptyset \) and \( \delta \in F_L(p^*(X)) \). Then \( \delta \) is said to be an \( L \)-Fuzzy hypersubset of \( X \), if for any finite family \( \{A_i\}_{i=1,2,...,n} \) of \( p^*(X) \) we have

\[ \delta \bigcup_{i=1}^n A_i \leq \bigvee_{i=1}^n \delta(A_i). \]

**Lemma 2.5.** Let \( \delta \in F_L(X) \). Then \( \delta \) induces an \( L \)-Fuzzy hypersubset \( \delta' \) of \( X \).

**Sketch of proof.** Define \( \delta' \in F_L(p^*(X)) \) as follows:

\[ \delta'(A) = \bigvee_{a \in A} \delta(a), \forall A \in p^*(X). \]

**Definition 2.6.** Let \( \mu \in F_L(p^*(X) \times p^*(X)) \). Then \( \mu \) is called an \( L \)-Fuzzy hyperrelation on \( X \) if:

\[ \mu \left( \bigcup_{i=1}^n E_i, \bigcup_{j=1}^m F_j \right) = \bigvee_{i=1}^n \bigwedge_{j=1}^m \mu(E_i, F_j) \]

for any finite families \( \{E_i\}_{i=1,2,...,n} \) and \( \{F_j\}_{j=1,2,...,m} \) of \( p^*(X) \).

**Remark 2.7.** Let \( \mu \) be an \( L \)-Fuzzy hyperrelation on \( X \). Then \( A \subseteq B \) and \( C \subseteq D \) imply that \( \mu(A, C) \leq \mu(B, D) \).

**Theorem 2.8.** Let \( \mu \) be an \( L \)-Fuzzy relation on \( X \). Then \( \mu \) induces an \( L \)-Fuzzy hyperrelation \( \mu' \) on \( X \).

**Sketch of proof.** Define \( \mu' \in F_L(p^*(X) \times p^*(X)) \) as follows:

\[ \mu'(A, B) = \bigvee_{a \in A, b \in B} \mu(a, b), \forall A, B \in p^*(X). \]

**Definition 2.9.** Let \( \delta \) be an \( L \)-Fuzzy hypersubset of \( X \) and \( \mu \) be an \( L \)-Fuzzy hyperrelation on \( X \). Then \( \mu \) is said to be a \((POM)_L\)-Fuzzy hyperrelation on \( \delta \) if

\[ \mu(E, F) \leq T(\delta(E), \delta(F)), \forall E, F \in p^*(X). \]
Lemma 2.10. Let \( \mu \) be a \((POM)_L\)-Fuzzy relation on \( \delta \), and \( \mu', \delta' \) be such as defined in Theorem 2.8 and Lemma 2.5 respectively. Then \( \mu' \) is a \((POM)_L\)-Fuzzy hyperrelation on \( \delta' \).

Proof. The proof is easy. \( \square \)

Definition 2.11. Let \( X \neq \emptyset \) and \( \xi = \{\mu_i\}_{i=1,2,\ldots,n} \) be a family of non-trivial \( L \)-Fuzzy subsets of \( X \) and \( X = \bigcup_{i=1}^{n} \mu_i^* \). Then the \( L \)-Fuzzy hyperrelation \( \mu \) on \( X \) is called a \((POM)_L\)-Fuzzy hyperrelation on \( \xi \) if for all \( i, j \in \{1,2,\ldots,n\} \):

\[
\mu(A, B) \leq T(\bigvee_{x \in A} \mu_i(x), \bigvee_{y \in B} \mu_j(y)), \quad \forall A, B \in p^*(X), \ A \subseteq \mu_i^*, B \subseteq \mu_j^*.
\]

Theorem 2.12. Let \( X = \{x_1,x_2,\ldots,x_n\} \), \( \mu \in F_L(p^*(X) \times p^*(X)) \) and \( \delta \in F_L(p^*(X)) \). If \( \mu \) is a \((POM)_L\)-Fuzzy hyperrelation on \( \delta \), then there is a family \( \xi \) of non-trivial elements of \( F_L(X) \) such that \( \mu \) is a \((POM)_L\)-Fuzzy hyperrelation on \( \xi \).

Sketch of proof. For each \( 1 \leq i \leq n \), define \( \mu_i \) as follows:

\[
\mu_i : X \rightarrow L, \ \mu_i(x) = \begin{cases} 
\delta(\{x_i\}) & \text{if } \delta(\{x_i\}) \neq 0, x = x_i \\
1 & \text{if } \delta(\{x_i\}) = 0, x = x_i \\
0 & \text{if } x \neq x_i
\end{cases}
\]

Now let \( \xi = \{\mu_1,\mu_2,\ldots,\mu_n\} \). Then it can be checked that \( \mu \) is a \((POM)_L\)-Fuzzy hyperrelation on \( \xi \).

Theorem 2.13. Let \( \mu \) be a \((POM)_L\)-Fuzzy hyperrelation on a family \( \xi = \{\mu_i\}_{i=1,2,\ldots,n} \) of \( L \)-Fuzzy subsets of \( X \). Then there exists an \( L \)-Fuzzy hypersubset \( \delta \) of \( X \) such that \( \mu \) is a \((POM)_L\)-Fuzzy hyperrelation on \( \delta \).

Proof. Define the \( L \)-Fuzzy hypersubset \( \delta \) of \( X \) as follows:

\[
\delta : p^*(X) \rightarrow L \\
A \rightarrow \bigvee_{i=1}^{n}(\bigvee_{x \in A \cap \mu_i^*} \mu_i(x)).
\]
It is obvious that \( \delta \) is well-defined and since each \( \mu_i \) is non-trivial for \( i = 1, 2, \ldots, n \), we conclude that \( \delta \) is also non-trivial. First we show that \( \delta \) is an \( L \)-Fuzzy hyper-subset of \( X \). Let \( \{A_j\}_{j=1,2,\ldots,t} \) be a finite family of \( p^*(X) \), then

\[
\delta \left( \bigcup_{j=1}^{t} A_j \right) = \bigvee_{i=1}^{n} \left( \bigvee_{x \in \left( \bigcup_{j=1}^{t} A_j \right) \cap \mu_i^*} \mu_i(x) \right) = \bigvee_{i=1}^{n} \left( \bigvee_{x \in \bigcup_{j=1}^{t} (A_j \cap \mu_i^*)} \mu_i(x) \right) = \bigvee_{i=1}^{n} \left( \bigvee_{x \in (A_i \cap \mu_i^*)} \mu_i(x) \right)
\]

Now let \( E,F \in p^*(X) \), then

\[
\mu(E,F) = \mu(X \cap E, X \cap F) = \mu \left( \bigcup_{i=1}^{n} \mu_i^* \cap E, \bigcup_{j=1}^{n} \mu_j^* \cap F \right) = \mu \left( \bigcup_{i=1}^{n} \mu_i^* \cap E \right) \bigcup_{j=1}^{n} \mu_j^* \cap F \right) = \bigvee_{i=1}^{n} \mu \left( \bigvee_{x \in \mu_i^* \cap E} \mu_i(x) \right) \bigvee_{j=1}^{n} \mu_j \left( \bigvee_{y \in \mu_j^* \cap F} \mu_j(y) \right)
\]

\[
\leq T \left( \bigvee_{i=1}^{n} \mu_i \left( x \right) \mid_{x \in \mu_i^* \cap E}, \bigvee_{j=1}^{n} \mu_j \left( y \right) \mid_{y \in \mu_j^* \cap F} \right), \quad \text{by } (*)
\]

\[
= T \left( \delta(E), \delta(F) \right).
\]

**Definition 2.14.** Let \( X \neq \emptyset \) be a finite set. Then the triple \( H = (X, \delta, \mu) \) is called a \((POM)_L\)-Fuzzy graph on \( X \) if

(i) \( \delta \in F_L(X) \),

(ii) \( \mu \in F_L(X \times X) \) and \( \mu \) is a \((POM)_L\)-Fuzzy relation on \( \delta \).

Note that if \( L = [0,1] \subseteq \mathbb{R} \) and \( T = \min \), then a \((POM)_L\)-Fuzzy graph is also a fuzzy graph.
Remark 2.15. Let $H = (X, \delta, \mu)$ be a $(POM)_L$-Fuzzy graph. So $\mu(x, y) \leq T(\delta(x), \delta(y))$, for all $x, y \in X$. If $x \not\in \delta^*$, then

$$\mu(x, y) \leq T(\delta(x), \delta(y)) = T(0, \delta(y)) = 0 ; \forall y \in X$$

That is $\mu(x, y) = 0$. So $(x, y) \not\in \mu^*$ for all $y \in X$. Now if we put $Y = \delta^* \subseteq X$, then $(Y, \delta|_{Y \times Y}, \mu|_{Y \times Y})$ is a $(POM)_L$-Fuzzy graph, called the saturated $(POM)_L$-Fuzzy subgraph of $(X, \delta, \mu)$.

From now on we let all $(POM)_L$-Fuzzy graph $(X, \delta, \mu)$ to be the saturated $(POM)_L$-Fuzzy subgraph of itself, so that $\delta^* = X$.

Definition 2.16. Let $X \neq \emptyset$ be a finite set and $\mathcal{H} = (X, \{\mu_i\}_{i=1,2,\ldots,n}, \mu)$. Then $\mathcal{H}$ is called a $(POM)_L$-Fuzzy hypergraph on $X$ if $\mu$ is a $(POM)_L$-Fuzzy hyperrelation on $\{\mu_i\}_{i=1,2,\ldots,n}$.

Theorem 2.17. Every $(POM)_L$-Fuzzy graph on $X$, induces (naturally) a $(POM)_L$-Fuzzy hypergraph on $X$.

Proof. Let $(X, \delta, \mu)$ be a $(POM)_L$-Fuzzy graph where $\delta^* = X = \{x_1, x_2, \ldots, x_n\}$. We define $\delta_i$, for all $i = 1, 2, \ldots, n$ as follows:

$$\delta_i : X \rightarrow L , \delta_i(x) = \begin{cases} \delta(x_i) & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases}$$

we have $\delta_i^* = \{x_i\}$ and $X = \bigcup_{i=1}^{n} \delta_i^*$. Consider $\mu' \in F_L(p^*(X) \times p^*(X))$ as defined in Theorem 2.8. Now we claim that $(X, \{\delta_i\}_{i=1,2,\ldots,n}, \mu')$ is a $(POM)_L$-Fuzzy hypergraph on $X$. To see this, since $\delta(x_i) = \delta_i(x_i)$ for all $i = 1, 2, \ldots, n$ we have

$$\mu'(\delta_i^*, \delta_j^*) = \mu'(\{x_i\}, \{x_j\}) = \mu(x_i, x_j) \leq T(\delta(x_i), \delta(x_j)) = T(\delta_i(x_i), \delta_j(x_j)) = T(\bigvee_{x \in \delta_i^*} \delta_i(x), \bigvee_{y \in \delta_j^*} \delta_j(y)).$$

Thus $\mu$ is a $(POM)_L$-Fuzzy hyperrelation on $\{\delta_i\}_{i=1,2,\ldots,n}$, and the proof is complete. \hfill \Box

Theorem 2.18. Let $X = \{x_1, x_2,\ldots, x_n\}$ and $(X, \{\mu_i\}_{i=1,2,\ldots,n}, \mu)$ be a $(POM)_L$-Fuzzy hypergraph on $X$ such that $\mu_i^* = \{x_i\}$, for $i = 1, 2, \ldots, n$. Then $\mu$ induces a $(POM)_L$-Fuzzy graph on $X$.

Sketch of Proof. Define $\delta \in F_L(X)$ and $\mu' \in F_L(X \times X)$ as follows:

$$\delta : X \rightarrow L , \delta(x_i) = \mu_i(x_i), \forall i = 1, 2, \ldots, n$$

and

$$\mu'(x_i, x_j) = \mu(\{x_i\}, \{x_j\}), \forall x_i, x_j \in X.$$ 

Then the proof can be completed by some calculations.

Theorem 2.19. (i) Every (ordinary) graph is a $(POM)_L$-Fuzzy graph. (ii) Every (ordinary) hypergraph is a $(POM)_L$-Fuzzy hypergraph.
Sketch of proof. (i) Let $G = (X, E)$ be a graph. Define
\[ \delta : X \to L, \quad \delta(x) = 1, \text{ for all } x \in X \]
and
\[ \mu : X \times X \to L, \quad \mu(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E \\ 0 & \text{if } (x, y) \notin E. \end{cases} \]
Then we see that $(X, \delta, \mu)$ is a $\text{(POM)}_L$-Fuzzy hypergraph on $X$.

(ii) Let $\mathcal{H} = (X, \{E_i\}_{i=1,\ldots,n})$ be a hypergraph. Define $\mu_i = \chi_{E_i}$, for all $i = 1, 2, \ldots, n$. Then if $\mu$ is an arbitrary $L$-Fuzzy hyperrelation on $X$, we conclude that $(X, \{\mu_i\}_{i=1,\ldots,n}, \mu)$ is a $\text{(POM)}_L$-Fuzzy hypergraph on $X$.

3. Category of $\text{(POM)}_L$-Fuzzy hypergraphs

**Definition 3.1.** Let $(X, \{\mu_i\}_{i=1,\ldots,n})$ and $(Y, \{\delta_i\}_{i=1,\ldots,m})$ be two $\text{(POM)}_L$-Fuzzy hypergraphs. If 
\[ \alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\} \]
and $f : X \to Y$ be two functions such that

(i) $f(\mu_i) \subseteq \delta^*_{\alpha(i)}$, $i = 1, 2, \ldots, n$,

(ii) $\mu_i(x) \leq \delta_{\alpha(i)}(f(x))$, $i = 1, 2, \ldots, n$, $\forall x \in X$,

(iii) $\mu(E, F) \leq \delta(f(E), f(F))$, $\forall E, F \in p^*(X)$,

then $(f, \alpha)$ is called a homomorphism of $\text{(POM)}_L$-Fuzzy hypergraphs.

**Category of $\text{(POM)}_L$-Fuzzy hypergraphs ($\text{(POM)}_L - \text{FHG}_r$):**

In order to construct the category $\text{(POM)}_L - \text{FHG}_r$ of all $\text{(POM)}_L$-Fuzzy hypergraphs, we consider all $\text{(POM)}_L$-Fuzzy hypergraphs as the objects of this category and for any two objects $X = (X, \{\mu_i\}_{i=1,\ldots,n}, \mu)$ and $Y = (Y, \{\delta_i\}_{i=1,\ldots,m}, \delta)$, we define $\text{Hom}(X, Y)$ as follows:

\[ \text{Hom}(X, Y) = \{(f, \alpha) | (f, \alpha) \text{ is a homomorphism from } X \text{ to } Y\}. \]

Now let $X = (X, \{\mu_i\}_{i=1,\ldots,n}, \mu)$, $Y = (Y, \{\delta_i\}_{i=1,\ldots,m}, \delta)$, $Z = (Z, \{\nu_i\}_{i=1,\ldots,n}, \nu)$ be three $\text{(POM)}_L$-Fuzzy hypergraphs and let $(f, \alpha) : X \to Y$ and $(g, \beta) : Y \to Z$ be two homomorphisms of $\text{(POM)}_L$-Fuzzy hypergraphs. We define the composition of these homomorphisms by

\[ (g, \beta) \circ (f, \alpha) = (g \circ f, \beta \circ \alpha). \]

Then $(g \circ f, \beta \circ \alpha)$ is a homomorphism from $X$ to $Z$, because for all $i = 1, 2, \ldots, n$ we have

i) $g \circ f(\mu_i) = g(f(\mu_i)) \subseteq g(\delta^*_i) \subseteq \nu^*_{\beta \circ \alpha}(i)$

ii) $\mu_i(x) \leq \delta_{\alpha(i)}(f(x)) \leq \nu_{\beta \circ \alpha}(i)(g \circ f(x))$, $\forall x \in X$

iii) $\mu(E, F) \leq \delta(f(E), f(F)) \leq \nu(g(f(E)), g(f(F)))$, $\forall E, F \in p^*(X)$.

Now it is easy to check that $(\text{POM})_L - \text{FHG}_r$ has all properties of a category.

**Theorem 3.2.** Let $(f, \alpha) : (X, \{\mu_i\}_{i=1,\ldots,n}, \mu) \to (Y, \{\delta_i\}_{i=1,\ldots,m}, \delta)$ be a homomorphism of $\text{(POM)}_L$-Fuzzy hypergraphs. Then $(f, \alpha)$ is an isomorphism in $(\text{POM})_L - \text{FHG}_r$ if and only if
(i) \( \alpha \in S_n \), where \( S_n \) is the permutation group on \{1,2,\ldots,n\},

(ii) \( f \) is one to one and onto,

(iii) \( f(\mu_i^\alpha) = \delta_\alpha^{i(\alpha)} \), \( i = 1,2,\ldots,n \),

(iv) \( \mu_i(x) = \delta_\alpha^{i(\alpha)}(f(x)) \), \( i = 1,2,\ldots,n \), \( \forall x \in X \),

(v) \( \mu(E, F) = \delta(f(E), f(F)) \), \( \forall E, F \in p^\ast(X) \).

**Proof.** (\( \Rightarrow \)) Let \( (f, \alpha) \) be an isomorphism. Then

(i), (ii): There exists a morphism \( (g, \beta) \) in \( (POM)_L - FHG_r \) such that \( (g, \beta) \circ (f, \alpha) = (1_X, 1_{\{1,2,\ldots,n\}}) \) and \( (f, \alpha) \circ (g, \beta) = (1_Y, 1_{\{1,2,\ldots,n\}}) \). These show that \( g \circ f = 1_X \), \( f \circ g = 1_Y \), \( \beta \circ \alpha = 1_{\{1,2,\ldots,n\}} \). This means that \( f \) is bijective and \( \alpha \in S_n \); moreover \( m = n \).

(iii): Let \( i \in \{1,2,\ldots,n\} \) and \( j = \alpha(i) \). Then \( \beta(j) = 1 \). So

\[
\delta_j^\beta \subseteq \mu_{\beta(j)}^\alpha \Rightarrow f(g(\delta_j^\beta)) \subseteq f(\mu_{\beta(j)}^\alpha)
\]

\[
\Rightarrow \delta_j^\beta \subseteq f(\mu_{\beta(j)}^\alpha) \Rightarrow \delta_j^\beta \subseteq f(\mu_i^\alpha).
\]

On the other hand we have \( f(\mu_i^\alpha) \subseteq \delta_j^\beta \). Thus \( \delta_j^{\alpha(i)} = f(\mu_i^\alpha) \).

(iv): Let \( i \in \{1,2,\ldots,n\} \), \( x \in X \) and \( \alpha(i) = j \), \( f(x) = y \). Then

\[
\mu_i(x) = \delta_{\alpha(i)}(f(x)).
\]

Since \( \delta_j^\beta \subseteq \mu_{\beta(j)}^\alpha \) we get that \( \delta_{\alpha(i)}(f(x)) \leq \mu_i(x) \). Thus \( \mu_i(x) = \delta_{\alpha(i)}(f(x)) \).

(v): Let \( E, F \subseteq X \), and \( A = f(E) \), \( B = f(F) \). Thus \( g(A) = E \) and \( g(B) = F \). Since \( \delta(A, B) \leq \mu(g(A), g(B)) \) we conclude that

\[
\delta(f(E), f(F)) \leq \mu(E, F) \leq \delta(f(E), f(F)),
\]

and (v) is proved.

(\( \Leftarrow \)) Define \( g = f^{-1} \) and \( \beta = \alpha^{-1} \), first we show that \( (g, \beta) \) is a morphism in \( (POM)_L - FHG_r \) from \( (Y, \{\delta\}, \{d\}) \) into \( (X, \{\mu\}, \{\delta\}) \).

Note that since \( \alpha \) is bijective, we must have \( m = n \). Let \( j \in \{1,2,\ldots,n\} \). Then there exists \( i \in \{1,2,\ldots,n\} \) such that \( i = \beta(j) \). For \( i \) we have \( f(\mu_i^\alpha) = \delta_{\alpha(i)}^\beta \), by (iii) so \( \mu_i^\alpha = g(\delta_{\alpha(i)}^\beta) \), and hence \( g(\delta_j^\beta) = \mu_i^\alpha \). Thus condition (i) of Definition 3.1 holds.

Now let \( j \in \{1,2,\ldots,n\} \) and \( y \in Y \). Then there exists \( i \in \{1,2,\ldots,n\} \) and \( x \in X \) such that \( i = \beta(j) \) and \( x = g(y) \). Now from (iv) we get

\[
\mu_i(x) = \delta_{\alpha(i)}(f(x)) \Rightarrow \mu_{\beta(j)}(g(y)) = \delta_j(y).
\]

Hence, the condition (ii) of Definition 3.1 holds too. Let \( A, B \subseteq Y \). Then there exist \( E, F \subseteq X \) such that \( E = g(A) \) and \( F = g(B) \). For \( E, F \) we have

\[
\mu(E, F) = \delta(f(E), f(F)) \Rightarrow \delta(A, B) = \mu(g(A), g(B)).
\]

Hence \( (g, \beta) \) is a morphism in \( (POM)_L - FHG_r \). It is clear that \( (g, \beta) \circ (f, \alpha) = (f, \alpha) \circ (g, \beta) = (1,1) \). So \( (g, \beta) \) is an isomorphism. \( \square \)

**Definition 3.3.** Let \( X = (X, \delta, \mu) \), \( Y = (Y, \delta', \mu') \) be two \( (POM)_L \)-Fuzzy graphs. If \( f : X \longrightarrow Y \) be a function such that:

(i) \( \delta(x) \leq \delta'(f(x)), \forall x \in X \),

(ii) \( \mu(x, y) \leq \mu'(f(x), f(y)), \forall (x, y) \in X \times Y \),

then we say that \( f \) is a homomorphism from \( X \) to \( Y \).
Category of \((POM)_L\)-Fuzzy graphs \(((POM)_L - FG_r)\):

We construct the category \((POM)_L - FG_r\) of all \((POM)_L\)-Fuzzy graphs. The objects of this category are all \((POM)_L\)-Fuzzy graphs, and for any two objects \(X = (X, \delta, \mu), Y = (Y, \delta', \mu')\), we define \(\text{Hom}(X, Y)\) to be the set of all homomorphism from \(X\) into \(Y\). It is easy to see that \((POM)_L - FG_r\) has all properties of a category.

**Theorem 3.4.** In \((POM)_L - FG_r\), coproduct exists, for any finite family of objects.

**Proof.** Let \(\{(A_i, \delta_i, \mu_i)\}_{i \in I}\) be a finite family of objects of \((POM)_L - FG_r\). For \(\{A_i\}_{i \in I}\). It is well-known that \(\bigcup_{i \in I} A_i, \lambda_i\) is a coproduct in the category of sets, where by \(\bigcup_{i \in I} A_i\) we mean the set \(\{(a, i) | (a, i) \in \bigcup_{i \in I}(A_i \times \{i\}), a \in A_i\}\), and for each \(i \in I\),

\[
\lambda_i : A_i \rightarrow \bigcup_{i \in I} A_i, \quad \lambda_i(a) = (a, i).
\]

Now we define

\[
\delta : \bigcup_{i \in I} A_i \rightarrow L, \quad \delta((a, i)) = \delta_i(a) \quad \text{for all}(a, i),
\]

and

\[
\mu : \bigcup_{i \in I} A_i \times \bigcup_{i \in I} A_i \rightarrow L, \quad \mu((a, i), (b, j)) = \begin{cases} 
\mu_i(a, b) & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

Then it is easy to see that \(\mu\) is a \((POM)_L\)-Fuzzy relation on \(\delta\). So \(\bigcup_{i \in I} A_i, \delta, \mu\) is an object in \((POM)_L - FG_r\). Now we show that for each \(i \in I\), \(\lambda_i : (A_i, \delta_i, \mu_i) \rightarrow (\bigcup_{i \in I} A_i, \delta, \mu)\) is a morphism in \((POM)_L - FG_r\). To see this:

(i) \(\delta(\lambda_i(a)) = \delta(a, i) = \delta_i(a)\), for all \(a \in A_i\),

(ii) \(\mu(\lambda_i(a), \lambda_i(b)) = \mu((a, i), (b, i)) = \mu_i(a, b), \forall a, b \in A_i\).

Thus \(\lambda_i\) is a morphism.

Next we prove that \(\{(\bigcup_{i \in I} A_i, \delta, \mu), \{\lambda_i\}_{i \in I}\}\) is a coproduct for \(\{(A_i, \delta_i, \mu_i)\}_{i \in I}\).

Let \((S, \delta', \mu')\) be a \((POM)_L\)-Fuzzy graph and \(\{f_i : (A_i, \delta_i, \mu_i) \rightarrow (S, \delta', \mu')\}_{i \in I}\) be a family of morphisms in \((POM)_L - FG_r\).

Since \(\bigcup_{i \in I} A_i, \{\lambda_i\}_{i \in I}\) is the coproduct for \(\{A_i\}_{i \in I}\) in the category of sets, we conclude that there exists a unique morphism \(\psi : \bigcup_{i \in I} A_i \rightarrow S\) in the category of
sets, such that
\[ \psi \circ \lambda_i = f_i , \quad \forall i \in I . \]
Now it can be checked that, in fact \( \psi : \left( \bigcup_{i \in I} A_i, \delta, \mu \right) \rightarrow (S, \delta', \mu') \) is a morphism in \( (POM)_L - FG_r \). Moreover it is unique, and this makes commutative the following diagram:
\[ \begin{array}{c}
  \left( \bigcup_{i \in I} A_i, \delta, \mu \right) \\
  \downarrow \psi \\
  (S, \delta', \mu') \\
\end{array} \]
that is \( \left( \bigcup_{i \in I} A_i, \delta, \mu \right) \) is a coproduct. \( \square \)

**Category of fuzzy subsets** \((FS)\) [5]:

The objects in this category are pairs \((S, \delta)\), where \( S \) is a set and \( \delta \) is a fuzzy subset on \( S \). A morphism from \((S, \delta)\) to \((S', \delta')\) is an ordinary function \( f : S \rightarrow S' \) such that \( \delta(x) \leq \delta'(f(x)), \forall x \in S \). The identity associated with the object \((S, \delta)\) is the identity map on the set \( S \).

The composition of maps \( f : (S, \delta) \rightarrow (S', \delta') \) and \( g : (S', \delta') \rightarrow (S'', \delta'') \) is \( g \circ f : (S, \delta) \rightarrow (S'', \delta'') \) where \( g \circ f : S \rightarrow S'' \) and \( \delta(S) \leq \delta'(f(S)) \leq \delta''(g(f(S))) \) for all \( s \in S \).

**Lemma 3.5.** Let \( \{(A_i, S_i)\}_{i \in I} \) be a family of fuzzy subsets. Then the product of this family exists in \( FS \).

**Definition 3.6.** [8] Let \((L,T)\) be a partially ordered monoid such that
\[ \bigwedge_{\alpha \in I} T(x_{\alpha}, y_{\beta}) \leq T\left( \bigwedge_{\alpha \in I} x_{\alpha}, \bigwedge_{\beta \in J} y_{\beta} \right), \]
for any two families \( \{x_{\alpha}\}_{\alpha \in I}, \{y_{\beta}\}_{\beta \in J} \) of elements of \( L \). Then we say that \( T \) satisfies the meet-property.

**Theorem 3.7.** Let \((L,T)\) be a partially ordered monoid which satisfies the meet-property. Then the product exists for any finite family of objects in \((POM)_L - FG_r\).
Sketch of Proof. Let \( \{ (A_i, \delta_i, \mu_i) \}_{i \in I} \) be a finite family of objects of \((POM)_L-FG_r\). Consider the product \( \prod_{i \in I} A_i \) of \( \{ A_i \}_{i \in I} \) in the category of sets. Define
\[
\delta : \prod_{i \in I} A_i \longrightarrow L ; \quad \delta((a_i)_{i \in I}) = \bigwedge_{i \in I} \delta_i(a_i) \quad (1)
\]
and
\[
\mu : \prod_{i \in I} A_i \times \prod_{i \in I} A_i \longrightarrow L ; \quad \mu((a_i)_{i \in I}, (b_i)_{i \in I}) \longrightarrow \bigwedge_{i \in I} \mu_i(a_i, b_i) \quad (2)
\]
Now we show that \( \mu \) is a \((POM)_L\)-Fuzzy relation on \( \delta \). We have
\[
\mu((a_i)_{i \in I}, (b_i)_{i \in I}) = \bigwedge_{i \in I} \mu_i(a_i, b_i) ; \quad \text{by (2)}
\]
\[
\leq \bigwedge_{i \in I} T(\delta_i(a_i), \delta_i(b_i)) ; \quad \text{since \( \mu_i \) is \( L_t \)-fuzzy relation on \( \delta_i, \forall i \in I \)}
\]
\[
\leq T\left( \bigwedge_{i \in I} \delta_i(a_i), \bigwedge_{i \in I} \delta_i(b_i) \right) ; \quad \text{by meet property of \( T \)}
\]
\[
= T(\delta((a_i)_{i \in I}), \delta((b_i)_{i \in I})) ; \quad \text{by (1)}
\]
Thus \( \prod_{i \in I} A_i, \delta, \mu \) is an object in \((POM)_L-FG_r\).

By considering the family \( \{ \Pi_i : \prod_{i \in I} A_i, \delta, \mu \longrightarrow (A_i, \delta_i, \mu_i) \}_{i \in I} \) of morphism in \((POM)_L-FG_r\), it is not difficult to prove that \((\prod_{i \in I} A_i, \delta, \mu), \{ \Pi_i \}_{i \in I} \) is the product of \( \{ (A_i, \delta_i, \mu_i) \}_{i \in I} \) in \((POM)_L-FG_r\).

**Theorem 3.8.** There exists a full and faithful functor from \((POM)_L-FG_r\) to \((POM)_L-FHG_r\). Hence there exists an embedding from \((POM)_L-FG_r\) into \((POM)_L-FHG_r\).

**Proof.** Let \((X, \delta, \mu)\) be a \((POM)_L\)-Fuzzy graph, and \( A, B \in p^*(X) \). Define
\[
\mu'(A, B) = \bigvee_{a \in A} ( \bigvee_{b \in B} \mu(a, b) )
\]
Then
\[
\mu'(A, B) = \bigvee_{a \in A} ( \bigvee_{b \in B} \mu(a, b) )
\]
\[
\leq \bigvee_{a \in A} ( \bigvee_{b \in B} T(\delta(a), \delta(b)) )
\]
\[
\leq T(\bigvee_{a \in A} \delta(a), \bigvee_{b \in B} \delta(b)) , \text{ by (*)}
\]
So \((X, \{\delta\}, \mu')\) is a \((POM)_L\)-Fuzzy hypergraph. Now define
\[
F : L_t - FG_r \rightarrow L_t - FHG_r,
\]
\[
(X, \delta, \mu) \mapsto (X, \{\delta\}, \mu')
\]
\[
f \mapsto (f, 1)
\]
for any morphism \(f : (X, \delta, \mu) \rightarrow (Y, \lambda, \nu)\) in \((POM)_L - FG_r\), where \(1 : \{1\} \rightarrow \{1\}\) is the identity function.

We show that \(F\) is a functor.

(i) \(f(\delta^*) = f(X) \subseteq Y = \lambda^*\)
(ii) Since \(f\) is a morphism in \((POM)_L - FG_r\), then
\[
\delta(x) \leq \lambda(f(x)), \forall x \in X.
\]
(iii) \(\mu'(E, F) = \bigvee_{a \in E} \bigvee_{b \in F} \mu(a, b) \leq \bigvee_{a \in E} \bigvee_{b \in F} \nu(f(a), f(b)) = \nu'(f(E), f(F)).\)

Thus \((f, 1)\) is a morphism in \((POM)_L\)-Fuzzy hypergraph. Now if \(g : (Y, \lambda, \nu) \rightarrow (Z, \xi, \rho)\) be a morphism in \((POM)_L - FG_r\), then
\[
f(goF) = (gof, 1) = (gof, 1o1)
\]
\[
= (g, 1)o(f, 1), \text{ by Definition of } L_t - FHG_r
\]
\[
= F(g)oF(f).
\]

It is clear that
\[
F(1_{(X, \delta, \mu)}) = (1_X, 1) = 1_{F(X)}.
\]

So \(F\) is a (covariant) functor.

Now let \(\mathcal{X} = (X, \delta, \mu)\) and \(\mathcal{Y} = (Y, \lambda, \nu)\) be two arbitrary objects in \((POM)_L - FG_r\). Consider two arbitrary morphisms \(f, g\) form \(\mathcal{X}\) to \(\mathcal{Y}\) such that \(F(f) = F(g)\). Thus we have \((f, 1) = (g, 1)\), which implies that \(f = g\). That is \(F\) is a faithful functor.

Also for the given objects \(\mathcal{X}\) and \(\mathcal{Y}\), let \((f, \alpha)\) be an arbitrary morphism from \(F(\mathcal{X}) = (X, \delta, \mu')\) to \(F(\mathcal{Y}) = (Y, \lambda, \nu')\) in \((POM)_L - FHG_r\). Then \(f : X \rightarrow Y\) and \(\alpha = 1 : \{1\} \rightarrow \{1\}\) are two functions. Now it is easy to check that \(f\) is a morphism from \(\mathcal{X}\) to \(\mathcal{Y}\) in \((POM)_L - FG_r\), and moreover \(F(f) = (f, 1) = (f, \alpha)\). Thus \(F\) is a full functor.

**Definition 3.9.** Let \((H, \ast)\) be a hypergroup and \(\delta \in F_L(H)\). Then \((H, \ast, \delta)\) is called a \((POM)_L\)-Fuzzy subhypergroup of \(H\) if

(i) \(T(\delta(x), \delta(y)) \leq \bigvee_{a \in x \ast y} \{\delta(a)\}, \forall x, y \in H\)
(ii) \(\forall x, a \in H, \exists y \in H\) such that \(x \in a \ast y\) and \(T(\delta(x), \delta(a)) \leq \delta(y)\).

(iii) \(\forall x, a \in H, \exists z \in H\) such that \(x \in z \ast a\) and \(T(\delta(x), \delta(a)) \leq \delta(z)\).
Example 3.10. Let $A$ be a set of $n$ elements, say $\{a_1, a_2, \ldots, a_n\}$. Then $L = (p(A), \subseteq)$ is a complete lattice which is not a chain. If we consider $T$ as follows:

$$T : L \times L \longrightarrow L$$

$$(B, C) \longmapsto B \cap C$$

Then $(L, T)$ is a partially ordered monoid. Now let $H = \{1, 2, \ldots, n\}$. Define the hyperoperation $\circ$ on $H$ by

$$\circ : H \times H \longrightarrow P^*\{H\}$$

$$(i, j) \longmapsto \{i, j\}$$

Then it is easy to see that $(H, \circ)$ is a (commutative) hypergroup. Clearly $\delta : H \longrightarrow L$ is an $H$, $\delta$-hypergroup. Without loss of generality we always suppose that $\delta$ and $x$ are all $L$-Fuzzy subsets on $H$. Now we can check that $(H, \circ, \delta)$ is an $(POM)_L$-Fuzzy subhypergroup of $H$. Moreover we can show that for any $k \in N$, $k \leq n$, $(H_k, \circ, \delta)$ is a $(POM)_L$-Fuzzy subhypergroup of $H_k$.

Remark 3.11. Let $(H, *, \delta)$ be a $(POM)_L$-Fuzzy subhypergroup of $H$. If $x \in H$ and $x \not\in \delta^*$, i.e. $\delta(x) = 0$, then the conditions of Definitions 3.9 always hold. Thus without loss of generality we always suppose that $\delta^* = H$.

Category of $(POM)_L$-Fuzzy subhypergroups $((POM)_L - FHG_p)$:

The objects are all $(POM)_L$-Fuzzy subhypergroups. A morphism form $(H, *, \delta)$ to $(H', *, \delta')$ is a function $f : H \longrightarrow H'$, satisfies

(i) $f(x * y) = f(x) *' f(y), \forall x, y \in H$

(ii) $\delta(x) \leq \delta'(f(x)), \forall x \in H$.

Lemma and Definition 3.12 (see [9]). Let $\delta \in F_L(X)$. Define $\mu_\delta \in F_L(X \times X)$ as follows:

$$\mu_\delta(x, y) = T(\delta(x), \delta(y)), \forall (x, y) \in X \times Y.$$ 

Then $\mu_\delta$ is a $(POM)_L$-Fuzzy relation on $\delta$, and called the strong $(POM)_L$-Fuzzy relation on $X$.

Proof. The proof is obvious. $\square$

Theorem 3.12. There exists a functor from $(POM)_L - FHG_p$ to $(POM)_L - FG_r$.

Proof. Let $(H, *, \delta)$ be a $(POM)_L$-Fuzzy subhypergroup. Define $F((H, *, \delta)) = (H, \delta, \mu_\delta)$. By Lemma 3.12 $(H, \delta, \mu_\delta)$ is a $(POM)_L$-Fuzzy graph. Let $f : (A, *, \delta) \longrightarrow (B, o, \delta)$ be a morphism in $(POM)_L - FHG_p$. Define $F(f) = f$. We have $F(f) : (A, \delta, \mu_\delta) \longrightarrow (B, \delta, \mu_\delta')$ such that

i) $\delta(a) \leq \delta'(f(a)), \forall a \in A$

ii) $\mu_\delta(a, b) = T(\delta(a), \delta(b))$

$$\leq T(\delta'(f(a)), \delta'(f(b)))$$

$$= \mu_{\delta'}(f(a), f(b)).$$

Therefore $F(f)$ is a morphism in $(POM)_L - FG_r$. It is clear that $F(1_{(A, *, \delta)}) = 1_{(A, \delta, \mu_\delta)}$, and $F(gof) = F(g) \circ F(f)$. Hence $F$ is a functor. $\square$

Theorem 3.13. There exists a functor from $(POM)_L - FHG_p$ to $(POM)_L - FHG_r$.
Proof. The proof follows from Theorems 3.8 and 3.13. □

Theorem 3.14. Let $L$ be totally ordered. Then every $(POM)_L$-Fuzzy hypergraph, induces a $(POM)_L$-Fuzzy hypergroup.

Proof. Let $H = (X, \{\mu_i\}_{i=1,2,...,n}, \mu)$ be a $(POM)_L$-Fuzzy hypergraph. Define

$$o : p^*(X) \times p^*(X) \longrightarrow p^*(p^*(X))$$

by

$$o(A, B) = o(B, A) = \{ C \in p^*(X) | \bigvee_{i=1}^{n} \bigvee_{a \in A} \mu_i(a) \leq \bigvee_{i=1}^{n} \bigvee_{c \in C} \mu_i(c) \leq \bigvee_{i=1}^{n} \bigvee_{b \in B} \mu_i(b)$$

or

$$\bigvee_{i=1}^{n} \bigvee_{b \in B} \mu_i(b) \leq \bigvee_{i=1}^{n} \bigvee_{c \in C} \mu_i(c) \leq \bigvee_{i=1}^{n} \bigvee_{a \in A} \mu_i(a) \}.$$

Thus clearly $A, B \in AoB, \forall A, B \in p^*(X)$. (1)

Now we must show that $(p^*(X), o)$ is a commutative hypergroup, to see this let $A, Y \in p^*(X)$ and $U = V = Y$. By (1) we have $Y \in AoV$ and $Y = UoA$. Therefore by Lemma 1.5 we have

$$Ao \circ p^*(X) = p^*(X) \circ A = p^*(X), \quad \forall A \in p^*(X).$$

Let $A, B, C \in p^*(X)$. Then by considering the totally ordered property of $L$, it is not difficult to check that $(AoB) \circ C = A \circ (BoC)$. Hence $(p^*(X), o)$ is a commutative hypergroup.

Now define

$$\delta : p^*(X) \longrightarrow L; \quad \delta(A) = \bigvee_{i=1}^{n} \mu_i(a), \quad \forall A \in p^*(X).$$

We claim that $(p^*(X), o, \delta)$ is a $(POM)_L$-Fuzzy subhypergroup. Let $A, B \in p^*(X)$ such that $\delta(A) \leq \delta(B)$, we have

$$\inf_{D \in AoB} \{ \delta(D) \} = \inf_{\delta(A) \leq \delta(D) \leq \delta(B)} \{ \delta(D) \} \geq \delta(A)$$

$$= T(\delta(A), 1) \geq T(\delta(A), \delta(B))$$

So condition (i) of Definition 3.9 holds.

Since $B \in AoB = BoA$ and $T(\delta(A), \delta(B)) \leq \delta(A)$, so conditions (ii) and (iii) of Definition 3.9 hold too. Therefore $(p^*(X), o, \delta)$ is a $(POM)_L$-Fuzzy subhypergroup.

Note that since $X = \bigcup_{i=1}^{n} \mu_i^*$, hence $\delta^* = p^*(X)$. □

Question: Let $L$ be totally ordered. Then can the object function defined in Theorem 3.14 be completed to a functor?
References


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