Nonlinear Deflection Analysis of Electrostatic Micro-Actuators with Different Electrode and Beam Shapes

Ming-Hung Hsu

Abstract—The nonlinear pull-in behavior for different electrostatic micro-actuators is simulated in this work. The difficulty of nonlinear equation is overcome using the differential quadrature method and Wilson–Θ method. Several characteristics of different combination of shaped fixed-fixed beam and curved electrode are also observed to optimize the design in this paper. The nonlinear deflection of uniform actuator and non-uniform actuator solved using the differential quadrature method are efficient. The stresses are determined for this par electrostatic micro-actuator design. The effects of applied voltage, squeeze film force, external loading and residual axial loading on the behavior of the electrostatic actuator are investigated. It is needed to consider the squeeze film force and residual axial loading in the fixed-fixed micro-actuator design.

Index Terms—Micro-beam, microelectromechanical system, electrostatic, squeeze film force, differential quadrature method, residual axial loading.

I. INTRODUCTION

Due to the electro-mechanical coupling behavior, the problems of electrostatic sensor and electrostatic actuator have been extensively investigated. Beam type silicon fabricated electrostatic actuators have been widely applied in the microelectromechanical systems. The nonlinear pull-in behavior of an electrostatic micro-actuator was proposed by Petersen [1]. Osterberg et al. [2] analyzed the electrostatically deformed diaphragms using one-dimensional numerical model and three-dimensional model. A CoSolve-EM simulation algorithm was used to analyze the three-dimensional coupled electro-mechanics for microelectromechanical systems by Gilbert et al. [3]. The dynamic behavior of active joints for different electrostatic actuator designs was studied by Elwenspoek et al. [4]. The idea of curved electrode in order to achieve a better pull-in performance was proposed by Legtenberg et al. [5], [6]. The Rayleigh-Ritz method is used in these papers to calculate the static deflections of different electrostatic actuators. The cantilever beam model was proposed to realize the characteristics of large-displacement actuators. Hirai et al. [7]-[9] presented the deflection characteristics of electrostatic actuators with shaped modified electrodes and cantilevers. Wang [10] applied a feedback control for suppressing the vibration of actuator beam in an electrostatic actuator. To evaluate the coupling effect between the electrostatic force and the elastic deformation, the combination of an exterior boundary element method for electrostatics and a finite element method for elasticity was presented by Shi et al. [11]. Osterberg and Senturia [12] showed that the sharp instability phenomena of electrostatic pull-in behaviors for cantilever beam and fixed-fixed beam actuators were adopted to extract the material properties of microelectromechanical systems. Grettillat et al. [13] employed the three dimensional MEMCAD and FEM programs to simulate the dynamics of actuator with the effect of squeeze-film damping. In order to enlarge the limit of travel distance before pull-in of electrostatic actuators, the leveraged bending and strain-stiffening methods were proposed by Hung and Senturia [14]. For the purpose of extracting material properties, the measurement of pull-in voltage and capacitance-voltage together with 2-D simulations including the electrical effects of fringing fields and finite beam thickness were performed by Chan et al. [15]. Since the electrostatic micro-actuators can undergo large deformation under large applied voltages, a mixed-regime approach to combine linear and nonlinear theories was proposed by Li and Aluru [16]. The differential quadrature method (DQM) with easy to use mesh less technique is employed to analyze the nonlinear deflection of different types micro-actuators with different loads, beam shapes and electrode shapes in this work. The effects of the electrode and beam shapes on the static deflection of an electrostatic actuator are investigated. The change of electrode shape and beam shape of an electrostatic actuator is an effective method to vary the electrostatic force distribution in the micro-electrostatic actuator. Actuator designs are proposed for improving the pull-in disadvantage. DQM is employed to formulate the electrostatic field problems in a matrix form. The Chebyshev-Gauss-Lobatto point distribution on each actuator is utilized in this paper.

II. THE DIFFERENTIAL QUADRATURE METHOD

There are a number of solution techniques, such as Rayleigh-Ritz method, analytical method, Galerkin method, finite element method and boundary element method for the complicated beam problems. The DQM is also among these. The DQM has been extensively used to solve a variety of problems in different fields of science and engineering without the necessity of energy formulation. The DQM has been shown to be a powerful contender in solving initial and boundary value problems and has thus become an alternative to the existing methods. Bert et al. [17]-[22] analyzed the static and free vibration
of beams and rectangular plates using the DQM. Jang et al. [23] proposed the $\delta$ technique for the boundary conditions of the structure. The boundary points are chosen at a small distance $\delta$. The $\delta$ technique can apply to the double boundary conditions of plate and beam problems. The $\delta$ cannot be enlarged for solution accuracy. The solutions oscillate when the $\delta$ is too small. The use of $\delta$ at the boundary makes the matrix ill conditioned. There is another method for treating the boundary conditions; Wang and Bert [24] took the boundary conditions into account in the weighting coefficients. Malik and Bert [25] solved the free vibration of the plates and showed that the boundary conditions can be built into the differential quadrature weighting coefficients. In the formulation, the multiple boundary conditions are directly applied to the weighting coefficients and thus it is not necessary to select a nearby point. In other words the accuracy of the calculated results will be independent of the value of $\delta$-interval. Quan and Chang [26], [27] derived the weighting coefficients in a more explicit way. The explicit formulae are more convenient. The weighting coefficients can be obtained by multiplication of the inverse matrix. Hsu and Kuang [28] analyzed the nonlinear dynamic behavior of electrostatic curved electrode actuators using the DQM. The DQM has been widely used to solve the linear and nonlinear differential problems. The essence of the DQM is that the derivative of a function at a sample point can be approximated as a weighted linear summation of the functional values at all of the sampling points in the domain. Using this approximation, the differential equation is then reduced into a set of algebraic equations.

For a function $f(x)$, the DQM approximation for the $m$-th order derivative at the $i$-th sampling point is given by

$$\frac{d^m f(x)}{dx^m} \bigg|_{x=x_j} = \sum_{j=1}^{N} D^{(m)}_{ij} f(x_j) \quad \text{for } i = 1, 2, ..., N$$

where

$$M(x) = \prod_{j=1}^{N}(x-x_j),$$

$$M_i(x_j) = \prod_{j=1,j\neq i}^{N}(x_j-x_j) \quad \text{for } i = 1, 2, ..., N$$

Equation (2) is substituted into (1); the weighting coefficients are given as

$$D^{(m)}_{ij} = \frac{M_i(x_j)}{(x_j-x_i)M_j(x_i)} \quad \text{for } i, j = 1, 2, ..., N \text{ and } i \neq j$$

and

$$D^{(m)}_{i} = -\sum_{j=1,j\neq i}^{N} D^{(m)}_{ij} \quad \text{for } i = 1, 2, ..., N$$

Once the sampling points, i.e. $x_i$ for $i = 1, 2, ..., N$, are selected, the coefficients of the weighting matrix can be obtained from (3) and (4). Higher-order derivatives of weighting coefficients can also be directly calculated by matrix multiplication [24], [25], which can be expressed as

$$D^{(m+1)}_{ij} = \sum_{k=1}^{N} D^{(m)}_{ik} D^{(1)}_{kj} \quad \text{for } i, j = 1, 2, ..., N$$

The selection of sampling points always plays an important role in the solution accuracy of the DQM. The unequally spaced sampling points of each beam using the Chebyshev-Gauss-Lobatto distribution [18] in the present computation are distributed as

$$x_i = \frac{1}{2} \left( 1 - \cos \left( \frac{i-1}{N-1} \pi \right) \right) \quad \text{for } i = 1, 2, ..., N$$

The integrity and computational efficiency of the DQM in this problem will be demonstrated through a series of case studies.

### III. DEFLECTIONS OF THE ELECTROSTATIC ACTUATORS

The fixed-fixed beam actuator is suspended on the fixed electrode as shown in Fig. 1. $t_o$ denotes the thickness at the base of the actuator. $L$ is the length of the actuator. Load $P$ is the residual axial loading acting on the fixed end of the actuator. The value of $P$ is $\sigma b h_o$. $\sigma$ is the residual stress and $b_h$ is the beam width. Load $q(x)$ is acting on $x = 0 - L$ in the beam. An electrostatic force pulls the fixed-fixed beam actuator toward to the curved electrode. The electrostatic force is introduced by the applied voltage difference between the curved electrode and the actuator. Different electrode shapes have been proposed to improve the electrostatic force distribution and the deform shape of the actuator. When the external voltage $e$ is applied between the deformable beam and the fixed electrode, a position dependent electrostatic pressure is created to pull...
the deformable beam toward the ground electrode. This
electrostatic pressure is proportional to the inverse of the
square of the gap between them. After the voltage exceeds
the critical voltage, the fixed-fixed beam will be pulled into
electrode suddenly. The dielectric layer can also prevent
the short circuit. The electric fringing effects are ignored in
following analyses. The strain energy of the bended
actuator can be expressed as
\[ U^e = \frac{1}{2} \int_{0}^{1} E I \left( \frac{\partial^2 u}{\partial x^2} \right)^2 \, dx \]
where \( E \) is Young’s modulus of the actuator and \( u \) is
the deflection of the beam in the direction of z-axis. Because of
the coupling between mechanical and electrostatic effects,
the behavior of the electrostatic actuator appears more
complicated than that in the elastic field. The electrode
begins at the point \( x = L_1 \) and ends at the point \( x = L_2 \).
With considering the electrostatic force, the virtual work
\( \delta W^e \) done by the bended actuator can be derived as
\[ \delta W^e = \int_{0}^{1} \frac{\varepsilon_0 e^2 \beta^2}{2} \left( d + S(x) - \alpha \sin(\frac{\pi x}{L} \gamma_0) \gamma_0 - u \right)^2 \delta u \, dx \]
where \( e \) is the applied voltage, \( \varepsilon_0 \) is the dielectric constant
in the air, \( b \) is the width of the actuator and \( d \) is the
initial gap as shown in Fig. 1. The cross-sectional area of
the actuator is
\[ A(x) = b_0 f(x) L + \alpha \sin(\frac{\pi x}{L} \gamma_0) \gamma_0 \]
where \( \alpha \) is the constant. \( I(x) \) is the moment of inertia of the
cross-sectional area of the actuator, it is
\( I(x) = I_0 (L + \alpha \sin(\frac{\pi x}{L} \gamma_0) \gamma_0) \gamma_0 \) and \( I_0 = b_0 f_0^2 / 12 \). Shape
function \( S(x) \) describes the shape of curved electrode and
it is presented as \( S(x) = \delta \beta \sin(\frac{\pi x}{L} \gamma_0) \gamma_0 \). \( \delta \beta \) is the fixed
end gap distance of curved electrode at \( x = 0 \) and \( x = L \).
The electrode shape is varied with the value of \( \beta \) and \( \delta \).
Substituting (9) and (10) into the principle of the total potential
energy
\[ \delta W^e - \delta U^e = 0 \]
The deflection \( u(x) \) of the actuator can be described by
the following nonlinear equation
\[ \frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) + \frac{P d^2 u}{dx^2} = \]
\[ \frac{\varepsilon_0 e^2 \beta^2}{2} \left( H(x - L_1)H(L_2 - x) - q(z) \right) + \int_{0}^{b_0} \mathcal{P} \left( 1 - \frac{4y^2}{b_0^2} \right) dy \]
A large class of microelectromechanical systems must
operate at significant gas pressure. The pressure within the
air film squeezed between two moving plates is described by
the two dimensional Reynolds equation [29]. The
equation is derived from the Navier-Stokes equation under
assumption that inertial terms are negligible compared to
viscous terms, there is no pressure gradient through the
film and the flow in the direction perpendicular to the
plates is negligible [29], [30]. The effects of the gas
surrounding the movable component are critical to understand
the electromechanical behavior of the micro-actuator. Most previous works in the squeeze film damping problems were simulated using the finite difference method and
the finite element method [31]-[34]. The differential
quadrature approach for the squeezed film force problem is
investigated in this work. The dynamic deflection of a
fixed-fixed micro-actuator can be expressed as the nonlinear differential equations shown in (17) and (18),
where \( \mu \) denotes the air viscosity, \( P_a \) is the ambient
pressure, \( \mathcal{P} \) is the pressure around the ambient pressure, and \( t_o \) specifies the thickness at the root of the actuator. \( \mathcal{P} \)
is much less than the ambient pressure. \( \rho_a \) is the density of the air. \( \rho \) is the density of the material of the actuator. The damping force \( c_a \frac{\partial^2 \delta}{\partial z^2} \) is assumed for resistance to the transverse velocity of the actuator. The damping force \( c_a \frac{\partial^2 \delta}{\partial z^2} (E I \frac{\partial^2 u}{\partial t \partial z}) \) is assumed for the resistance to the strain velocity of the micro-actuator. \( e \) is the applied voltage, \( \varepsilon_0 \) is the dielectric constant of air, \( b_0 \) is the width of the actuator, and \( d \) is the initial gap. By employing the DQM, (1) is substituted into (12). Combining with the boundary conditions, the deflection equation of the actuator can be discretized with the sample points as

\[
[K][u] = [F] \quad (19)
\]

The elements in the stiffness matrices are

\[
K_{ii} = 1
\]

\[
K_{ij} = 0 \quad \text{for} \quad j = 2, 3, ..., N \quad (20)
\]

\[
K_{2j} = \frac{D^{(1)}_{j}}{L} \quad \text{for} \quad j = 1, 2, ..., N \quad (21)
\]

\[
K_{ij} = \frac{\frac{d^2}{dx^2}[EI(x)]}{2} + \frac{d^2}{dx^2}[EI(x)] \frac{D^{(1)}_{j}}{L} + EI(x) \frac{D^{(2)}_{j}}{L^2} \quad \text{for} \quad i = 3, 4, ..., N - 2 \quad \text{and} \quad j = 1, 2, ..., N \quad (22)
\]

\[
K_{N-1,i} = \frac{D^{(1)}_{i}}{L} \quad \text{for} \quad j = 1, 2, ..., N \quad (23)
\]

\[
K_{N,i} = 0 \quad \text{for} \quad j = 1, 2, ..., N - 1 \quad (24)
\]

\[
K_{N,N} = 1
\]

\[
F_i = 0 \quad \text{for} \quad i = 1, 2
\]

\[
F_i = \frac{e_r \varepsilon_0 e^2}{2} \left( d + S(x) - \frac{\alpha \sin(\pi x / L) b_0}{2} - u_i \right) \quad (25)
\]

\[
q \quad \text{for} \quad i = 3, 4, ..., N - 2
\]

\[
F_i = 0 \quad \text{for} \quad i = N - 1, N
\]

IV. NUMERICAL RESULTS AND DISCUSSION

Figure 2 shows the schematic view of a cantilever beam electrostatic actuator. Fig. 3 shows the tip deflections with different applied voltages and electrode shapes. The material and the geometric parameters of the cantilever actuator are: \( E = 150.0 \times 10^6 \) Pa, \( \delta_0 = 30.0 \mu m \), \( b_0 = 5.0 \mu m \), \( e_0 = 8.85 \times 10^{-12} \), \( t_0 = 2.0 \mu m \), \( d = 2.0 \mu m \), \( \mu = 1.8 \times 10^{-5} \text{Ns/m}^2 \) and \( L = 500.0 \mu m \). The shape of the electrode varied with \( S(x) \). The results imply that the tip deflections calculated from the proposed DQM are very consistent with the experimental results published in the literature [5], [6]. Fig. 4 compares the displacement of the actuator with varying applied voltage. The material of the actuator is polysilicon. The numerical results display that the displacement of the beam is sensitively when applied voltage greater than 875 V. Numerical results in this example show that the driving voltage can affect the electromechanical behavior of the actuator system significantly. Calculated results also display the higher driving voltage applied between electrodes always introduce larger actuator deflection. Fig. 5 implies the stress of the beam with different applied voltage. The max stress of the actuator approaches near the fixed end. There is a significant change when applied voltage greater than 875 V. Figs. 6 and 7 show the comparison of the deflections of the actuators with different \( \alpha \) and \( \beta \). The shapes of the electrode and beam varied with \( \alpha \) and \( \beta \).

The electrode and beam shapes effects able affect the mechanical behavior of the actuator. Fig. 8 compares the
deflections of the actuators under various loads \( q(x) \). The displacement of deflectable actuator increases as the load \( q(x) \) decreases. Fig. 9 displays the deflections of the actuator with different residual axial loading \( P \). Numerical results indicated that the stiffness of the taper actuator is increased for the actuator with a larger value of residual axial loading \( P \). Residual axial loading should be considered in the design. Numerical results indicate that the DQM is a feasible and efficient method to analyze the nonlinear pull-in behavior of a fixed-fixed type electrostatic micro-beam. Fig. 10 shows the deflections near the middle of the electrostatic fixed-fixed actuator. The applied voltage is 620 V. Numerical results demonstrate that the micro-actuator design needs to consider squeeze film force effect. Maximum displacements occur near the middle of the fixed-fixed actuator. The deflections with squeeze film force are smaller than the deflections with no squeeze film force. Fig. 11 shows the stresses near the base of the electrostatic fixed-fixed actuator. Maximum stresses appear near the root of the electrostatic actuator. The stresses without squeeze film force are greater than the stresses with squeeze film force. The squeeze film damping diminishes the dynamic response of the fixed-fixed micro-actuator.
micro-actuators is simulated using the DQM. Numerical film damping effects and residual axial loading effects in system are investigated. It is essential to consider squeeze and residual axial loading in a micro-electrostatic-actuator.

The electrostatic behavior of the fixed-fixed beam type micro-actuators is simulated using the DQM. Numerical results indicate that DQM can provide efficient deflection estimation for different type electrostatic micro-actuators. The effects of electrode and fixed-fixed beam shapes on the deflection of the deflectable actuator with external loading and residual axial loading in a micro-electrostatic-actuator system are investigated. It is essential to consider squeeze film damping effects and residual axial loading effects in the design. The DQM are quite suitable for designing and analyzing an electrostatic micro-actuator.

V. CONCLUSION

The electrostatic behavior of the fixed-fixed beam type micro-actuators is simulated using the DQM. Numerical results indicate that DQM can provide efficient deflection estimation for different type electrostatic micro-actuators. The effects of electrode and fixed-fixed beam shapes on the deflection of the deflectable actuator with external loading and residual axial loading in a micro-electrostatic-actuator system are investigated. It is essential to consider squeeze film damping effects and residual axial loading effects in the design. The DQM are quite suitable for designing and analyzing an electrostatic micro-actuator.

REFERENCES


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