Nonlinear Deflection Analysis of Electrostatic Micro-Actuators with Different Electrode and Beam Shapes

Ming-Hung Hsu

Abstract—The nonlinear pull-in behavior for different electrostatic micro-actuators is simulated in this work. The difficulty of nonlinear equation is overcome using the differential quadrature method and Wilson-θ method. Several characteristics of different combination of shaped fixed-fixed beam and curved electrode are also observed to optimize the design in this paper. The nonlinear deflection of uniform actuator and non-uniform actuator solved using the differential quadrature method are efficient. The stresses are determined for this par electrostatic micro-actuator design. The effects of applied voltage, squeeze film force, external loading and residual axial loading on the behavior of the electrostatic actuator are investigated. It is needed to consider the squeeze film force and residual axial loading in the fixed-fixed micro-actuator design.

Index Terms—Micro-beam, microelectromechanical system, electrostatic, squeeze film force, differential quadrature method, residual axial loading.

I. INTRODUCTION

Due to the electro-mechanical coupling behavior, the problems of electrostatic sensor and electrostatic actuator have been extensively investigated. Beam type silicon fabricated electrostatic actuators have been widely applied in the microelectromechanical systems. The nonlinear pull-in behavior of an electrostatic micro-actuator was proposed by Petersen [1]. Osterberg et al. [2] analyzed the electrostatically deformed diaphragms using one-dimensional numerical model and three-dimensional model. A CoSolve-EM simulation algorithm was used to analyze the three-dimensional coupled electro-mechanics for microelectromechanical systems by Gilbert et al. [3]. The dynamic behavior of active joints for different electrostatic actuator designs was studied by Elwenspoek et al. [4]. The idea of curved electrode in order to achieve a better pull-in performance was proposed by Legtenberg et al. [5], [6]. The Rayleigh-Ritz method is used in these papers to calculate the static deflections of different electrostatic actuators. The cantilever beam model was proposed to realize the characteristics of large-displacement actuators. Hirai et al. [7]-[9] presented the deflection characteristics of electrostatic actuators with shaped modified electrodes and cantilevers. Wang [10] applied a feedback control for suppressing the vibration of actuator beam in an electrostatic actuator. To evaluate the coupling effect between the electrostatic force and the elastic deformation, the combination of an exterior boundary element method for electrostatics and a finite element method for elasticity was presented by Shi et al. [11]. Osterberg and Senturia [12] showed that the sharp instability phenomena of electrostatic pull-in behaviors for cantilever beam and fixed-fixed beam actuators were adopted to extract the material properties of microelectromechanical systems. Gretillat et al. [13] employed the three dimensional MEMCAD and FEM programs to simulate the dynamics of actuator with the effect of squeeze-film damping. In order to enlarge the limit of travel distance before pull-in of electrostatic actuators, the leveraged bending and strain-stiffening methods were proposed by Hung and Senturia [14]. For the purpose of extracting material properties, the measurement of pull-in voltage and capacitance-voltage together with 2-D simulations including the electrical effects of fringing fields and finite beam thickness were performed by Chan et al. [15]. Since the electrostatic micro-actuators can undergo large deformation under large applied voltages, a mixed-regime approach to combine linear and nonlinear theories was proposed by Li and Aluru [16].

The differential quadrature method (DQM) with easy to use mesh less technique is employed to analyze the nonlinear deflection of different types micro-actuators with different loads, beam shapes and electrode shapes in this work. The effects of the electrode and beam shapes on the static deflection of an electrostatic actuator are investigated. The change of electrode shape and beam shape of an electrostatic actuator is an effective method to vary the electrostatic force distribution in the micro-electrostatic actuator. Actuator designs are proposed for improving the pull-in disadvantage. DQM is employed to formulate the electrostatic field problems in a matrix form. The Chebyshev-Gauss-Lobatto point distribution on each actuator is utilized in this paper.

II. THE DIFFERENTIAL QUADRATURE METHOD

There are a number of solution techniques, such as Rayleigh-Ritz method, analytical method, Galerkin method, finite element method and boundary element method for the complicated beam problems. The DQM is also among these. The DQM has been extensively used to solve a variety of problems in different fields of science and engineering without the necessity of energy formulation. The DQM has been shown to be a powerful contender in solving initial and boundary value problems and has thus become an alternative to the existing methods. Bert et al. [17]-[22] analyzed the static and free vibration
of beams and rectangular plates using the DQM. Jang et al. [23] proposed the $\delta$ technique for the boundary conditions of the structure. The boundary points are chosen at a small distance $\delta$. The $\delta$ technique can apply to the double boundary conditions of plate and beam problems. The $\delta$ cannot be enlarged for solution accuracy. The solutions oscillate when the $\delta$ is too small. The use of $\delta$ at the boundary makes the matrix ill conditioned. There is another method for treating the boundary conditions; Wang and Bert [24] took the boundary conditions into account in the weighting coefficients. Malik and Bert [25] solved the free vibration of the plates and showed that the boundary conditions can be built into the differential quadrature weighting coefficients. In the formulation, the multiple boundary conditions are directly applied to the weighting coefficients and thus it is not necessary to select a nearby point. In other words the accuracy of the calculated results will be independent of the value of $\delta$-interval. Quan and Chang [26], [27] derived the weighting coefficients in a more explicit way. The explicit formulae are more convenient. The weighting coefficients can be obtained by multiplication of the inverse matrix. Hsu and Kuang [28] analyzed the nonlinear dynamic behavior of electrostatic curved electrode actuators using the DQM. The DQM has been widely used to solve the linear and nonlinear differential problems. The essence of the DQM is that the derivative of a function at a sample point can be approximated as a weighted linear summation of the functional values at all of the sampling points in the domain. Using this approximation, the differential equation is then reduced into a set of algebraic equations.

For a function $f(x)$, the DQM approximation for the $m$-th order derivative at the $i$-th sampling point is given by

$$
\frac{d^m f(x)}{dx^m} \bigg|_{x=x_j} = \sum_{j=1}^{N} D^{(m)}_{i,j} f(x_j) \quad \text{for } i = 1, 2, \ldots, N \tag{1}
$$

where $f(x_j)$ is the functional value at the sample point $x_j$, and $D^{(m)}_{i,j}$ are the weighting coefficients of the $m$-th order differentiation attached to these points. A test function is introduced to overcome the numerical ill-conditions in determining the weighting coefficients $D^{(m)}_{i,j}$ [26], [27], that is

$$
f(x) = \sum_{i=1}^{N} M(x) f(x_j) \quad \text{for } i = 1, 2, \ldots, N \tag{2}
$$

where

$$
M(x) = \prod_{j=1}^{N} (x - x_j),
$$

$$
M_i(x_j) = \prod_{j=1, j \neq i}^{N} (x_j - x_j) \quad \text{for } i = 1, 2, \ldots, N
$$

Equation (2) is substituted into (1); the weighting coefficients are given as

$$
D^{(m)}_{i,j} = \frac{M_i(x_j)}{(x_i - x_j)M_j(x_i)} \quad \text{for } i, j = 1, 2, \ldots, N \text{ and } i \neq j \tag{3}
$$

and

$$
D^{(1)}_{i,j} = -\sum_{k=1}^{N} D^{(1)}_{i,j'} f(x_j') \quad \text{for } i = 1, 2, \ldots, N \tag{4}
$$

Once the sampling points, i.e. $x_i$ for $i = 1, 2, \ldots, N$, are selected, the coefficients of the weighting matrix can be obtained from (3) and (4). Higher-order derivatives of weighting coefficients can also be directly calculated by

Holistic analysis of the solution accuracy of the DQM. The equally spaced sampling points of each beam using the Chebyshev-Gauss-Lobatto distribution [18] in the present computation are distributed as

$$
x_j = \frac{1}{2} \left( 1 - \cos \left( \frac{i-1}{N-1} \pi \right) \right) \quad \text{for } i = 1, 2, \ldots, N \tag{8}
$$

The integrity and computational efficiency of the DQM in this problem will be demonstrated through a series of case studies.

III. DEFLECTIONS OF THE ELECTROSTATIC ACTUATORS

The fixed-fixed beam actuator is suspended on the fixed electrode as shown in Fig. 1. $t_0$ denotes the thickness at the base of the actuator. $L$ is the length of the actuator. Load $P$ is the residual axial loading acting on the fixed end of the actuator. The value of $P$ is $\sigma h_0 b$, $\sigma$ is the residual stress and $b_0$ is the beam width. Load $q(x)$ is acting on $x = 0 \sim L$ in the beam. An electrostatic force pulls the fixed-fixed beam actuator toward the curved electrode. The electrostatic force is introduced by the applied voltage difference between the curved electrode and the actuator. Different electrode shapes have been proposed to improve the electrostatic force distribution and the deform shape of the actuator. When the external voltage $e$ is applied between the deformable beam and the fixed electrode, a position dependent electrostatic pressure is created to pull
the deformable beam toward to the ground electrode. This electrostatic pressure is proportional to the inverse of the square of the gap between them. After the voltage exceeds the critical voltage, the fixed-fixed beam will be pulled into electrode suddenly. The dielectric layer can also prevent the short circuit. The electric fringing effects are ignored in following analyses. The strain energy of the bended actuator can be simplified as

\[ U^e = \frac{1}{2} \int_0^L E \left( \frac{\partial^2 u}{\partial x^2} \right)^2 \, dx - \frac{1}{2} \int_0^L \rho \left( \frac{\partial u}{\partial t} \right)^2 \, dx \]  

(9)

where \( E \) is Young’s modulus of the actuator and \( u \) is the deflection of the beam in the direction of x-axis. Because of the coupling between mechanical and electrostatic effects, the behavior of the electrostatic actuator appears more complicated than that in the elastic field. The electrode begins at the point \( x = L_1 \) and ends at the point \( x = L_2 \). With considering the electrostatic force, the virtual work \( \delta W^e \) done by the bended actuator can be derived as

\[ \delta W^e = \int_{L_1}^{L_2} \varepsilon_0 \varepsilon_b e^2 \left( d + S(x) - \frac{\alpha \sin(\pi x / L) y_0}{2} - u \right)^2 \delta u \, dx \]

(10)

\[ - \int_0^L q(x) \delta u \, dx \]

where \( \varepsilon_0 \) is the applied voltage, \( \varepsilon_b \) is the dielectric constant in the air, \( b_0 \) is the width of the actuator and \( d \) is the initial gap as shown in Fig. 1. The cross-sectional area of the actuator is \( A(x) = b_0 \int_0^x (1 + \alpha \sin(\pi x / L)) \), where \( \alpha \) is the constant. \( I(x) \) is the moment of inertia of the cross-sectional area of the actuator, \( I(x) = I_0 (1 + \alpha \sin(\pi x / L)) \) and \( I_0 = b_0^3 / 12 \). Shape function \( S(x) \) describes the shape of curved electrode and it is presented as \( S(x) = \delta + \beta \sin(\pi x / L) \). \( \delta \) is the fixed end gap distance of curved electrode at \( x = 0 \) and \( x = L \). The electrode shape is varied with the value of \( \beta \) and \( \delta \). Substituting (9) and (10) into the principle of the total potential energy

\[ \delta W^e - \delta U^e = 0 \]

(11)

The deflection \( u(x) \) of the actuator can be described by the following nonlinear equation

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) + \frac{d}{dx} \left( -\frac{\alpha \sin(\pi x / L) y_0}{2} + \rho \frac{\partial u}{\partial t} + c_w \frac{\partial u}{\partial t} + c_w \frac{\partial^2 u}{\partial t^2} \right) \frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) =
\]

\[
\varepsilon_0 \varepsilon_b e^2 \left( H(x - L_1)H(L_2 - x) \right)^2 \left( q(z) + \int_{x/2}^{b_0} \overline{P} \left( 1 - \frac{4y^2}{b_0^2} \right) dy \right)
\]

\[
\frac{\partial}{\partial x} \left( \rho \left( d + S(x) - \frac{\alpha \sin(\pi x / L) y_0}{2} - u \right)^2 \right) \frac{\partial}{\partial x} \left( P + \overline{P} \left( 1 - \frac{4y^2}{b_0^2} \right) \right) + \frac{\partial}{\partial y} \left( \rho \left( d + S(x) - \frac{\alpha \sin(\pi x / L) y_0}{2} - u \right)^2 \right)
\]

\[
\frac{\partial}{\partial y} \left( \rho \left( d + S(x) - \frac{\alpha \sin(\pi x / L) y_0}{2} - u \right)^2 \right) \frac{\partial}{\partial y} \left( P + \overline{P} \left( 1 - \frac{4y^2}{b_0^2} \right) \right)
\]

(12)

where \( H(x) \) is the Heaviside step function. The corresponding boundary conditions of the clamped-clamped actuator are

\[ u = 0 \quad \text{at} \quad x = 0 \]

(13)

\[ \frac{du}{dx} = 0 \quad \text{at} \quad x = 0 \]

(14)

\[ u = 0 \quad \text{at} \quad x = L \]

(15)

\[ \frac{du}{dx} = 0 \quad \text{at} \quad x = L \]

(16)

A large class of microelectromechanical systems must operate at significant gas pressure. The pressure within the air film squeezed between two moving plates is described by the two dimensional Reynolds equation [29]. The equation is derived from the Navier-Stokes equation under assumption that inertial terms are negligible compared to viscous terms, there is no pressure gradient through the film and the flow in the direction perpendicular to the plates is negligible [29], [30]. The effects of the gas surrounding the movable component are critical to understand the electromechanical behavior of the micro-actuator. Most previous works in the squeeze film damping problems were simulated using the finite difference method and the finite element method [31]-[34]. The differential quadrature approach for the squeezed film force problem is investigated in this work. The dynamic deflection of a fixed-fixed micro-actuator can be expressed as the nonlinear differential equations shown in (17) and (18), where \( \mu \) denotes the air viscosity, \( P \) is the ambient pressure, \( \overline{P} \) is the pressure around the ambient pressure, and \( t_0 \) specifies the thickness at the root of the actuator. \( \overline{P} \)
is much less than the ambient pressure. \( \rho_a \) is the density of the air. \( \rho \) is the density of the material of the actuator. The damping force \( c_v \partial u / \partial t \) is assumed for resistance to the transverse velocity of the actuator. The damping force \( c_v \partial^2 z / \partial t^2 (EI \partial^2 u / \partial t \partial z) \) is assumed for the resistance to the strain velocity of the micro-actuator. \( e \) is the applied voltage, \( \epsilon_0 \) is the dielectric constant of air, \( b_0 \) is the width of the actuator, and \( d \) is the initial gap. By employing the DQM, (1) is substituted into (12). Combining with the boundary conditions, the deflection equation of the actuator can be discretized with the sample points as

\[
[K][u] = [F]
\]

The elements in the stiffness matrices are

\[
K_{ii} = 1
\]

\[
K_{ij} = 0 \quad \text{for} \quad j = 2, 3, \ldots, N
\]

\[
K_{ij} = \frac{D_{ij}^{(i)}}{L} \quad \text{for} \quad j = 1, 2, \ldots, N
\]

\[
K_{yy} = \frac{d_1^2}{dx^2} \left[ EI(x) \right]_{x=x_0} \frac{D_{yy}^{(i)}}{L} + 2 \frac{d}{dx} \left[ EI(x) \right]_{x=x_0} \frac{D_{yy}^{(i)}}{L} + EI(x) \frac{D_{yy}^{(i)}}{L} + P \frac{D_{yy}^{(i)}}{L^2}
\]

for \( i = 3, 4, \ldots, N - 2 \) and \( j = 1, 2, \ldots, N \)

\[
K_{N-1,i} = \frac{D_{N-1}^{(i)}}{L} \quad \text{for} \quad j = 1, 2, \ldots, N
\]

\[
K_{N,j} = 0 \quad \text{for} \quad j = 1, 2, \ldots, N - 1
\]

\[
K_{NN} = 1
\]

\[
F_i = 0 \quad \text{for} \quad i = 1, 2
\]

\[
F_i = \frac{e_r \rho \epsilon \alpha \epsilon \alpha^2}{2} \left( d + S(x) - \frac{\alpha \sin(\pi x / L) t_0 - \alpha}{2} \right)
\]

\[
-q \quad \text{for} \quad i = 3, 4, \ldots, N - 2
\]

\[
F_i = 0 \quad \text{for} \quad i = N - 1, N
\]

IV. NUMERICAL RESULTS AND DISCUSSION

Figure 2 shows the schematic view of a cantilever beam electrostatic actuator. Fig. 3 shows the tip deflections with different applied voltages and electrode shapes. The material and the geometric parameters of the cantilever actuator are: \( E = 150.0 \text{ GPa} \), \( \delta_0 = 30.0 \mu \text{m} \), \( b_0 = 5.0 \mu \text{m} \), \( \epsilon_r = 8.85 \times 10^{-12} \), \( \epsilon_0 = 2.0 \mu \text{m} \), \( d = 2.0 \mu \text{m} \), \( \mu = 1.8 \times 10^{-4} \text{ Ns/m}^2 \) and \( L = 500.0 \mu \text{m} \). The shape of the electrode varies with \( S(x) \). The results imply that the tip deflections calculated from the proposed DQM are very consistent with the experimental results published in the literature [5], [6]. Fig. 4 compares the displacement of the actuator with varying applied voltage. The material of the actuator is polysilicon. The numerical results display that the displacement of the beam is sensitively when applied voltage greater than 875 V. Numerical results in this example show that the driving voltage can affect the electromechanical behavior of the actuator system significantly. Calculated results also display the higher driving voltage applied between electrodes always introduce larger actuator deflection. Fig. 5 implies the stress of the beam with different applied voltage. The stress of the actuator approaches near the fixed end. There is a significant change when applied voltage greater than 875 V. Figs. 6 and 7 show the comparison of the deflections of the actuators with different \( \alpha \) and \( \beta \). The shapes of the electrode and beam varied with \( \alpha \) and \( \beta \).

The electrode and beam shapes effects able affect the mechanical behavior of the actuator. Fig. 8 compares the
deflections of the actuators under various loads $q(x)$. The displacement of deflectable actuator increases as the load $q(x)$ decreases. Fig. 9 displays the deflections of the actuator with different residual axial loading $P$. Numerical results indicated that the stiffness of the taper actuator is increased for the actuator with a larger value of residual axial loading $P$. Residual axial loading should be considered in the design. Numerical results indicate that the DQM is a feasible and efficient method to analyze the nonlinear pull-in behavior of a fixed-fixed type electrostatic micro-beam. Fig. 10 shows the deflections near the middle of the electrostatic fixed-fixed actuator. The applied voltage is 620 V. Numerical results demonstrate that the micro-actuator design needs to consider squeeze film force effect. Maximum displacements occur near the middle of the fixed-fixed actuator. The deflections with squeeze film force are smaller than the deflections with no squeeze film force. Fig. 11 shows the stresses near the base of the electrostatic fixed-fixed actuator. Maximum stresses appear near the root of the electrostatic actuator. The stresses without squeeze film force are greater than the stresses with squeeze film force. The squeeze film damping diminishes the dynamic response of the fixed-fixed micro-actuator.
results indicate that DQM can provide efficient deflection micro-actuators is simulated using the DQM. Numerical estimation for different type electrostatic micro-actuators.

analyzing an electrostatic micro-actuator.

film damping effects and residual axial loading effects in a micro-electrostatic-actuator deflection of the deflectable actuator with external loading.

V. CONCLUSION

The electrostatic behavior of the fixed-fixed beam type micro-actuators is simulated using the DQM. Numerical results indicate that DQM can provide efficient deflection estimation for different type electrostatic micro-actuators. The effects of electrode and fixed-fixed beam shapes on the deflection of the deflectable actuator with external loading and residual axial loading in a micro-electrostatic-actuator system are investigated. It is essential to consider squeeze film damping effects and residual axial loading effects in the design. The DQM are quite suitable for designing and analyzing an electrostatic micro-actuator.

REFERENCES


Ming-Hung Hsu was born in Taiwan. He received his Doctor of Philosophy degree in Mechanical and Electromechanical Engineering from the National Sun Yat-Sen University, in 2003. He is currently an associate professor in the Department of Electrical Engineering at National Penghu University. He has published over 60 journal and conference papers. His research interests include electric machines, mechanical vibration and microelectromechanical systems.