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CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

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ABSTRACT. In this paper, some results of Singh, Gopalakrishna and Kulkarni (1970s) have been extended to higher order derivatives. It has been shown that, if \( \sum a \Theta(a, f) = 2 \) holds for a meromorphic function \( f(z) \) of finite order, then for any positive integer \( k \), \( T(r, f) \sim T(r, f^{(k)}), r \to \infty \) if \( \Theta(\infty, f) = 1 \) and \( T(r, f^{(k)}) \sim (k + 1)T(r, f), r \to \infty \) if \( \Theta(\infty, f) = 0 \).

1. Introduction

Let \( f(z) \) be a meromorphic function in the complex plane \( \mathbb{C} \). Assume that basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [3], [10] or [12]) are known. We use the following notations of frequent use of value distribution (see [3]) with their usual meaning:

\[ m(r, f), N(r, a), \overline{N}(r, a), \delta(a, f), \Theta(a, f), \cdots \]

As usual, if \( a = \infty \), we write \( N(r, \infty) = N(r, f), \overline{N}(r, \infty) = \overline{N}(r, f) \). We denote by \( S(r, f) \) any quantity such that

\[ S(r, f) = o(T(r, f)), \quad r \to +\infty \]

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without restriction if \( f(z) \) is of finite order and otherwise except possibly for a set of values of \( r \) of finite linear measure. The well known Nevanlinna’s deficiency relation states that
\[
\sum_a \delta(a, f) \leq \sum_a \Theta(a, f) \leq 2.
\]

If \( \sum_a \delta(a, f) = 2 \), then we say that \( f(z) \) has maximum deficiency sum (see [2]).

Let \( f(z) \) be a meromorphic scalar valued function in \( \mathbb{C} \). On the characteristic function of derivative of \( f(z) \) with maximum deficiency sum has been studied by Shan, Singh, Gopalakrishna, Edrei and Weitsman [1], [4]-[8]. For example, Edrei [1] and Weitsman [8] have proved

**Theorem A** Let \( f(z) \) be a transcendental meromorphic function of finite order and assume \( \sum \delta(a, f) = \eta \geq 1 \) and \( \delta(\infty) = 2 - \eta \). Then
\[
T(r, f') \sim \eta T(r, f), r \to +\infty.
\]

If \( \sum_a \delta(a, f) = 2 \) is replaced by \( \sum \Theta(a, f) = 2 \), Singh, Gopalakrishna [6] and Singh, Kulkarni [7] have proved

**Theorem B** Let \( f(z) \) be a transcendental meromorphic scalar valued function of finite order and assume \( \sum \Theta(a, f) = 2 \). Then
\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty).
\]

Hence

1. if \( \Theta(\infty, f) = 1 \), \( T(r, f) \sim T(r, f') \) as \( r \to \infty \);
2. if \( \Theta(\infty, f) = 0 \), \( T(r, f') \sim 2T(r, f) \) as \( r \to \infty \).

We extend the above result to higher order derivatives as follows:

**Theorem 1.1.** Suppose that \( f \) is a transcendental meromorphic function of finite order and \( \sum \Theta(a, f) = 2 \). Then for any positive integer \( k \), we have

1. if \( \Theta(\infty, f) = 1 \), \( T(r, f) \sim T(r, f^{(k)}) \) as \( r \to \infty \);
(2) if $\Theta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k + 1)T(r, f)$ as $r \to \infty$.

From Theorem 1.1, we can get

**Corollary 1.2.** [11] Suppose that $f$ is a transcendental meromorphic function of finite order and $\sum a = 2$. Then for any positive integer $k$, we have

1. if $\delta(\infty, f) = 1$, $T(r, f) \sim T(r, f^{(k)})$ as $r \to \infty$;

2. if $\delta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k + 1)T(r, f)$ as $r \to \infty$.

2. Proof of Theorem 1.1

**Proof.** (1) We prove Theorem 1.1 (1) by induction. Since $\Theta(\infty, f) = 1$, by Theorem B, we have $T(r, f) \sim T(r, f')$ as $r \to \infty$. Assume that

\[ T(r, f) \sim T(r, f^{(k)}), \quad r \to \infty. \]  

(2.1)

Now we prove $T(r, f) \sim T(r, f^{(k+1)})$ as $r \to \infty$.

Without loss of generality we can assume that $q \geq 2$. Put

\[ F(z) = \sum_{i=1}^{q} \frac{1}{f(z) - a_i}, \quad a_i \in \mathbb{C}. \]

Then (See [6])

\[ \sum_{i=1}^{q} m(r, a_i) \leq m(r, F) + O(1). \]

So

\[ \sum_{i=1}^{q} m(r, a_i) \leq m(r, F) + O(1) \]

\[ = m \left( r, \frac{1}{f^{(k+1)}}f^{(k+1)} \right) + O(1) \]

\[ \leq m \left( r, \frac{1}{f^{(k+1)}} \right) + m \left( r, \sum_{i=1}^{q} \frac{f^{(k+1)}(z) - a_i}{f(z) - a_i} \right) + O(1) \]

\[ = m \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f). \]
Hence
\[
qT(r, f) \leq \sum_{i=1}^{q} N(r, a_i) + m \left( r, \frac{1}{f(k+1)} \right) + S(r, f)
\]
\[
= \sum_{i=1}^{q} N(r, a_i) + T \left( r, f^{(k+1)} \right) - N \left( r, \frac{1}{f(k+1)} \right) + S(r, f)
\]
\[
\leq \sum_{i=1}^{q} N(r, a_i) + T \left( r, f^{(k+1)} \right) - N \left( r, \frac{1}{f'} \right) + S(r, f)
\]
\[
= T \left( r, f^{(k+1)} \right) + \sum_{i=1}^{q} N(r, a_i) - N_0 \left( r, \frac{1}{f'} \right) + S(r, f).
\]

where \( N_0 \left( r, \frac{1}{f'} \right) \) is formed with the zeros of \( f' \) which are not zeros of any of the \( f - a_i, i = 1, 2, \cdots, q \). Since \( N_0 \left( r, \frac{1}{f'} \right) \geq 0 \), we have
\[
qT(r, f) \leq T \left( r, f^{(k+1)} \right) + \sum_{i=1}^{q} N(r, a_i) + S(r, f).
\]

Thus
\[
\sum_{i=1}^{q} \left( 1 - \frac{N(r, a_i)}{T(r, f)} \right) \leq \frac{T \left( r, f^{(k+1)} \right)}{T(r, f)} + \frac{S(r, f)}{T(r, f)}.
\]

So
\[
\sum_{i=1}^{q} \Theta(a_i, f) \leq \lim inf_{r \to \infty} \frac{T \left( r, f^{(k+1)} \right)}{T(r, f)},
\]
holds for any \( q \geq 2 \). Letting \( q \to \infty \), we obtain
\[
(2.2) \quad 1 = \sum_{a \neq \infty} \Theta(a, f) \leq \lim inf_{r \to \infty} \frac{T \left( r, f^{(k+1)} \right)}{T(r, f)}.
\]

Combining (2.1) and (2.2) we have
\[
(2.3) \quad 1 \leq \lim inf_{r \to \infty} \frac{T \left( r, f^{(k+1)} \right)}{T(r, f)} \leq \lim sup_{r \to \infty} \frac{T \left( r, f^{(k+1)} \right)}{T(r, f(k))}.
\]
On the other hand, since $N(r, f^{(k)}) = N(r, f), \Theta(\infty, f) = 1$ and (2.1), we have
\[
\limsup_{r \to \infty} \frac{N(r, f^{(k)})}{T(r, f^{(k)})} \leq \limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)} = 0.
\]
So
\[
\Theta(\infty, f^{(k)}) = 1.
\]
Thus
\[
T(r, f^{(k+1)}) = m(r, f^{(k+1)}) + N(r, f^{(k+1)}) \leq m(r, f^{(k)}) + N(r, f^{(k+1)}) + N(r, f^{(k)}) + S(r, f).
\]
Hence
\[
(2.4) \quad \limsup_{r \to \infty} \frac{T(r, f^{(k+1)})}{T(r, f^{(k)})} \leq 2 - \Theta(\infty, f^{(k)}) = 1.
\]
(2.1) and (2.3)-(2.4) together imply $T(r, f) \sim T(r, f^{(k+1)})$ as $r \to \infty$.

We can prove (2) of Theorem 1.1 by using the same method as that in [11]. As [11] may not be abundantly available, we give the following proof. From Nevanlinna’s second fundamental theorem, we have
\[
(q - 1)T(r, f) \leq T(r, f) + \sum_{i=1}^{q} \overline{N}(r, a_i) + N(r, f) + S(r, f).
\]
Thus
\[
\sum_{i=1}^{q} \Theta(a_i, f) \leq 1 + \liminf_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2.
\]
Letting $q \to \infty$, we obtain
\[
2 = \sum_{a \neq \infty} \Theta(a, f) \leq 1 + \liminf_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2.
\]
So
\[
(2.5) \quad T(r, f) \sim N(r, f) \sim \overline{N}(r, f), r \to \infty.
\]
Since
\[(k + 1)\overline{N}(r, f) \leq N(r, f) + k\overline{N}(r, f) = N\left(r, f^{(k)}\right)\]
\[\leq T\left(r, f^{(k)}\right)\]
\[\leq m(r, f) + m\left(r, \frac{f^{(k)}}{f}\right) + N\left(r, f^{(k)}\right)\]
\[= T(r, f) + k\overline{N}(r, f) + S(r, f).\]
From this and (2.5), we get \(T(r, f^{(k)}) \sim (k + 1)T(r, f)\) as \(r \to \infty\). \(\square\)

3. Proof of Corollary 1.2

Proof. Since \(\delta(a, f) \leq \Theta(a, f)\) for every \(a \in \mathbb{C} \cup \{\infty\}\), if \(\sum a \delta(a, f) = 2\), then \(\sum a \Theta(a, f) = 2\) and \(\delta(a, f) = \Theta(a, f)\) for every \(a \in \mathbb{C} \cup \{\infty\}\). Hence Corollary 1.2 follows by Theorem 1.1. \(\square\)

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