COMPLEMENT OF SPECIAL CHORDAL GRAPHS AND VERTEX DECOMPOSABILITY

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Communicated by Ebadollah S. Mahmoodian

ABSTRACT. In this paper, we introduce a subclass of chordal graphs which contains $d$-trees and show that their complement are vertex decomposable and so is shellable and sequentially Cohen-Macaulay. This result improves the main result of Ferrarello who used a theorem due to Fröberg and extended a recent result of Dochtermann and Engström.

1. Introduction

Let $k$ be a field. To any finite simple graph $G$ with vertex set $V = [n] := \{1, \cdots, n\}$ and edge set $E(G)$ one associates an ideal $I(G) \subset k[x_1, \cdots, x_n]$ generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. The ideal $I(G)$ and the quotient ring $k[x_1, \cdots, x_n]/I(G)$ are called the edge ideal of $G$ and the edge ring of $G$, respectively. The independence complex of $G$ is defined by

$$\text{Ind}(G) = \{A \subseteq V | A \text{ is an independent set in } G\},$$

$A$ is said to be an independent set in $G$ if none of its elements are adjacent. Note that $\text{Ind}(G)$ is precisely the simplicial complex with the Stanley-Reisner ideal $I(G)$. We denoted by $\Delta_G$ the clique complex of $G$, which is the simplicial complex with vertex set $V$ and with faces

Keywords: sequentially Cohen-Macaulay, vertex decomposable, chordal graphs.
Received: 8 March 2011, Accepted: 25 May 2011.
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the cliques of $G$. It is easy to see that $\Delta_G = \text{Ind}(\overline{G})$, where $\overline{G}$ is the complement of $G$.

A simplicial complex $\Delta$ is recursively defined to be vertex decomposable if it is either a simplex, or has some vertex $v$ so that:

- both $\Delta \setminus v$ and $\text{link}_\Delta v$ are vertex decomposable, and
- no face of $\text{link}_\Delta v$ is a facet of $\Delta \setminus v$.

A vertex $v$ which satisfies the second condition is called a shedding vertex.

A simplicial complex $\Delta$ is called shellable if the facets (maximal faces) of $\Delta$ can be ordered a $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$, cf [1]. The notion of shellability was discovered in the context of convex polytopes, cf. [8].

The dimension of a face $F$ is $|F| - 1$. Let $d = \max\{|F| : F \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim(\Delta) = d - 1$. A simplicial complex is pure if all of its facets are of the same dimension. The k-skeleton of $\Delta$ is the complex generated by all the k-dimensional faces of $\Delta$. A complex is sequentially Cohen-Macaulay if its k-skeleton is Cohen-Macaulay for each k, $k < \text{dimension of the complex}$. Any shellable complex is sequentially Cohen-Macaulay.

We have the following chains of strict implications:

- vertex decomposable $\overset{(i)}{\implies}$ shellable $\overset{(ii)}{\implies}$ sequentially Cohen-Macaulay
- pure vertex decomposable $\overset{(iii)}{\implies}$ pure shellable $\overset{(iv)}{\implies}$ Cohen-Macaulay

Where $(i)$ comes from [12, Lemma 6], $(ii)$ from [10, p. 87], $(iii)$ from [9, Theorem 2.8] and $(iv)$ from [7].

In recent years there have been a flurry of work investigating how the combinatorial properties of $G$ appear within the algebraic properties of $R/I(G)$, and vice versa. (Sequentially) Cohen-Macaulay rings are of great interest. As a consequence, one particular stream of research has focused on the question of what graph $G$ has the property that $R/I(G)$ is (Sequentially) Cohen-Macaulay.

We can recursively define a generalized $d$-tree in the following way:

1. A complete graph with $d+1$ vertices is a generalized $d$-tree;
2. Let $G$ be a graph on the vertex set $V(G)$. Suppose that there is some vertex $v \in V(G)$ such that the followings hold:
   (i) the restriction $G_1$ of $G$ to $V_1 = V \setminus \{v\}$ is a generalized $d$-tree;
(ii) there is a subset $V_2$ of $V_1$, where the restriction of $G$ to $V_2$ is a clique of size $j$ with $0 \leq j \leq d$;
(iii) $G$ is the graph generated by $G_1$ and the complete graph on $V_2 \cup \{v\}$.

In particular, we say that $G$ is a $(d, j)$-tree if in the above recursive definition $j$ is fixed. A $(d, d)$-tree is called a $d$-tree.

A graph $G$ is called chordal if every cycle of length $> 3$ has a chord. Recall that a chord of a cycle is an edge which joins two vertices of the cycle but is not itself an edge of the cycle. In [2] Dirac proved that the generalized d-trees are exactly the chordal graphs and so $(d, j)$-trees are chordal.

Many authors are interested in the case when $G$ or its complement is chordal (in particular $d$-tree) for example see [4], [5], and sections 3 and 4 of [3].

Ferrarello in [4] showed that the complement of a $d$-tree is Cohen-Macaulay and Dochtermann and Engström in [3] extended this result by showing that the complement of a $d$-tree is pure shellable.

It is not hard to show that the complement of a $d$-tree is pure of dimension $d$ [see Proposition 2.4], but in general the complement of a chordal graph is not pure so it is natural to ask whether the complement of a chordal graph is sequentially Cohen-Macaulay. By giving an example we show that the answer is negative. We show that if $G$ is the complement of a $(d, j)$-tree, then $\text{Ind}(G)$ is vertex decomposable and so shellable and sequentially Cohen-Macaulay. This result is a generalization of [4, Theorem 3.3] and [3, Proposition 3.6].

2. Main Results

The definition of vertex decomposable complexes translates nicely to independence complexes as follows:

Lemma 2.1. [13, Lemma 2.2]. An independence complex $\text{Ind}(G)$ is vertex decomposable if $G$ is a totally disconnected graph (with no edges), or if

(i) $G \setminus v$ and $G \setminus N[v]$ are both vertex decomposable, and
(ii) An independent set in $G \setminus N[v]$ is not a maximal independent set in $G \setminus v$.

We say that a graph $G$ is vertex decomposable if its independence complex $\text{Ind}(G)$ is vertex decomposable. Let $N(v)$ denotes the open
neighborhood of \( v \), that is, all vertices adjacent to \( v \). Let \( N[v] \) denotes the close neighborhood of \( v \), which is \( N(v) \) together with \( v \) itself, so that \( N[v] = N(v) \cup \{v\} \).

**Theorem 2.2.** Let \( G \) be the complement of a \((d, j)\)-tree, then \( \text{Ind}(G) \) is vertex decomposable and so it is shellable and sequentially Cohen-Macaulay.

**Proof.** We use induction on \( |V(G)| \). If \( |V(G)| = d + 1 \), then \( G \) is totally disconnected and there is nothing to prove.

Let the statement be true for complement of \((d, j)\)-trees of size \( < n \) and let \( G \) be a \((d, j)\)-tree with \( |V(G)| = n \).

It is easy to see that there is a vertex \( v \) in \( G \) such that \( G[N_G(v)] \) is a clique and \( G[N_G(v)] \) is not a maximal clique in \( G \setminus v \). Now \( G \setminus N[v] \) is a totally disconnected graph, so it is a vertex decomposable graph. On the other hand \( G \setminus v \) is a chordal graph so by the induction hypothesis \( G \setminus v \) is vertex decomposable.

Now it suffices to show that no independent set in \( G \setminus N[v] \) is a maximal independent set in \( G \setminus v \), but it holds obviously, because \( G[N_G(v)] \) is not a maximal clique in \( G \setminus v \).

\( \square \)

Note that in general it is not the case that the complement of a chordal graph is vertex decomposable. See the following example:

**Example 2.3.** Let \( G \) be the graph in figure 1:

![Figure 1](image)

Here \( G \) is a chordal graph but \( \overline{G} \) is bipartite and does not have any end vertex (a vertex of degree 1). Thus by [11, Lemma 3.9] \( \overline{G} \) is not sequentially Cohen-Macaulay and so it is not vertex decomposable.
By using a result of Fröberg, Ferrarello in [4] showed that the complement of a d-tree is Cohen-Macaulay and Dochtermann and Engström in [3] extended this result by showing that the complement of a d-tree is pure shellable. We state the following result as a generalization:

**Corollary 2.4.** Let $G$ be the complement of a d-tree, then $\text{Ind}(G)$ is pure vertex decomposable (pure shellable and Cohen-Macaulay).

**Proof.** Using Theorem 2.2, it remains to prove the purity. The facets of $\text{Ind}(G)$ are the maximal independent sets of $G$, which are in fact the maximal cliques of $\overline{G}$. So it suffices to show that every maximal clique of a d-tree, $H$, is of size $d+1$. We use induction on $|V(H)|$, if $|V(H)| = d + 1$ then $H = K_{d+1}$, and there is nothing to prove. Let the statement be true for every d-tree of size $< n$ and let $|V(H)| = n$. It is easy to check that there is a vertex in $V(H)$, say $v$, such that $\deg v = d$ and $H[N[v]]$ is a clique. Let $C$ be a maximal clique in $H$ if $v \notin C$ then $C$ is a maximal clique in $H \setminus v$, so by the induction hypothesis, $|C| = d + 1$. And if $v \in C$ the desired statement holds obviously. □

In the following we give some examples of (pure and non-pure) vertex decomposable graphs.

By using the mentioned theorem of Dirac, one can show that chordal graphs are vertex decomposable, see [3] and [13]. The following example shows that the converse is not true in general.

**Example 2.5.** Our first example is $T = P_n$ (a path with $n$ vertices). When $n \geq 5$, $\overline{T}$ is pure vertex decomposable but not chordal. (Note that the complement of a tree $T$ is chordal if and only if $\text{diam}(T) \leq 3$, where $\text{diam}(G) = \max\{d(u,v) | u, v \in V(G)\}$.

The complement of a tree $T$ is chordal if and only if $\text{diam}(T) \leq 3$. In the following example we show that this is not true for $d$-trees when $d > 1$.

**Example 2.6.** The graph $G$ in Figure 2 is a 2-tree with diameter 2. $\overline{G}$ is pure vertex decomposable and has an induced cycle of length 4.
Example 2.7. The graph $G$ in Figure 3 is a $(3, 2)$-tree which is not a $d$-tree. Therefore $\overline{G}$ is vertex decomposable but not pure.

References

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