SEQUENTIALY COHEN-MACaulAY GRAPHS OF FORM $\theta_{n_1, \ldots, n_k}$

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Abstract. Let $k$ be an integer greater than 2 and $n_1, \ldots, n_k$ be a sequence of positive integers with at most one of them being equal to 1. Let $\theta_{n_1, \ldots, n_k}$ be a graph consisting of $k$ paths, having only their endpoints in common. We characterize all sequentially Cohen-Macaulay graphs of this type. We also show for these types of graphs the notions of vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent.

1. Introduction

Let $G$ be a finite simple graph. To $G$ with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E(G)$, one can associate an ideal $I(G) \subset R = K[x_1, \ldots, x_n]$, called the edge ideal of $G$, which is generated by all monomials $x_ix_j$ such that $\{i, j\} \in E(G)$. Here, $K$ is an arbitrary field. The independence complex $\Delta_G$ of a graph $G$ is defined by

$$\Delta_G = \{A \subseteq V | A \text{ is an independent set in } G\},$$

where $A$ is an independent set in $G$ if none of its elements are adjacent. Note that $\Delta_G$ is precisely the simplicial complex associated with $I(G)$.

It is a well-known consequence of Menger’s Theorem [5, Theorem 3.3.5] that each 3-connected graph has an induced subgraph of the form

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θ_{p,q,r}, for some natural numbers p, q and r. This was our motivation to study sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \).

A graded \( R \)-module \( M \) is called *sequentially Cohen-Macaulay* (over \( K \)) if there exists a finite filtration of graded \( R \)-modules,

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,
\]
such that each \( M_i/M_{i-1} \) is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing; that is,

\[
\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).
\]

A graph \( G \) is said to be sequentially Cohen-Macaulay, if \( R/I(G) \) is a sequentially Cohen-Macaulay \( R \)-module.

On the other hand, a simplicial complex \( \Delta \) is called *shellable*, in the sense of Björner and Wachs [1], if the facets (maximal faces) of \( \Delta \) can be ordered as \( F_1, \ldots, F_s \) such that for all \( 1 \leq i < j \leq s \), there exists some \( v \in F_j \setminus F_i \) and some \( l \in \{1, \ldots, j-1\} \) with \( F_j \setminus F_l = \{v\} \). A graph \( G \) is called shellable, if \( \Delta_G \) is a shellable simplicial complex. In [12], Stanley showed that every shellable simplicial complex was sequentially Cohen-Macaulay, but the converse was not true.

Studying shellable or sequentially Cohen-Macaulay graphs has attracted significant attentions of researchers working in the borderline of combinatorial commutative algebra and algebraic combinatorics; see [1, 6, 7, 8, 10, 14, 16]. In [8], Francisco and Van Tuyl characterized all sequentially Cohen-Macaulay cycles. They showed that the \( n \)-cycle \( C_n \) was sequentially Cohen-Macaulay if and only if \( n \in \{3, 5\} \) (see [8, Proposition 4.1]). In [6], Faridi showed that simplicial trees were sequentially Cohen-Macaulay. Moreover, in [10], sequentially Cohen-Macaulay cacti graphs (a cactus is a connected graph in which each edge belongs to at most one cycle) were characterized. In addition, in [14], Van Tuyl and Villarreal showed that a bipartite graph \( G \) was shellable if and only if it was sequentially Cohen-Macaulay (see [14, Theorem 3.8]).

Here, we determine all sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \), where \( \{n_1, \ldots, n_k\} \neq \{2, 5\} \). For \( \{n_1, \ldots, n_k\} \neq \{2, 5\} \), we show in Theorem 2.6 that \( \theta_{n_1, \ldots, n_k} \) is sequentially Cohen-Macaulay if and only if \( \{1, 2\} \subseteq \{n_1, \ldots, n_k\} \) or \( \{2, 3\} \subseteq \{n_1, \ldots, n_k\} \) or \( \{n_1, \ldots, n_k\} = \{1, 4\} \). Moreover, as a result of this theorem, in Theorem 2.7 we show those graphs of the form \( \theta_{n_1, \ldots, n_k} \), which satisfy each one of the latter relations, are sequentially Cohen-Macaulay if and only if they are shellable or vertex decomposable.
Finally, in Proposition 2.8, we show that for \( \{n_1, \ldots, n_k\} = \{2, 5\} \), the graph \( \theta_{n_1, \ldots, n_k} \) is not vertex decomposable. Therefore, we characterize all vertex decomposable graphs of the form \( \theta_{n_1, \ldots, n_k} \) in Theorem 2.9. In Proposition 2.10, by direct computation, we show that for \( k = 3 \) and \( \{n_1, \ldots, n_k\} = \{2, 5\} \), the graph \( \theta_{n_1, \ldots, n_k} \) is not even sequentially Cohen-Macaulay. This result and computational evidences from some other examples lead us to conjecture that all graphs of the form \( \theta_{n_1, \ldots, n_k} \), for which \( \{n_1, \ldots, n_k\} = \{2, 5\} \), are not sequentially Cohen-Macaulay.

Characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \) with [13, Lemma 2.4] and [14, Theorem 2.9] enable us to get more examples of vertex decomposable, shellable and sequentially Cohen-Macaulay graphs.

2. Sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \)

Let \( k \) be an integer greater than 1 and \( n_1, \ldots, n_k \) be a sequence of positive integers. Let \( \theta_{n_1, \ldots, n_k} \) be the graph constructed by \( k \) paths of length \( n_1, \ldots, n_k \), with only their endpoints being in common. By length of a path, we mean the number of edges in the path. Since the graphs are assumed simple, at most one of the \( n_i \)’s in \( \theta_{n_1, \ldots, n_k} \) can be equal to one. If \( k = 2 \), then \( \theta_{n_1, \ldots, n_k} \) would be a cycle of length \( n_1 + n_2 \). The vertex decomposable and sequentially Cohen-Macaulay graphs of these types are completely studied in [8, 16]. Here, we assume \( k > 2 \) and characterize all vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \).

Given a simplicial complex \( \Delta \) on \([n]\), the Alexander dual complex \( \Delta^\vee \) is defined by \( \Delta^\vee = \{[n] \setminus F | F \notin \Delta\} \). Unless otherwise stated, when we discuss the Alexander dual \( \Delta^\vee \) of a simplicial complex \( \Delta \), we assume that \( [n] \setminus i \notin \Delta \), for all \( i \in [n] \). Thus, \( \Delta^\vee \) is again a simplicial complex on \([n]\).

Let \( I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t}) \) be a square-free monomial ideal. The ideal
\[
I^\vee = (x_{1,1}, \ldots, x_{1,s_1}) \cap \cdots \cap (x_{t,1}, \ldots, x_{t,s_t})
\]
is called the Alexander dual of \( I \). These two ideals are related in the following way. If \( I \) is the Stanley-Reisner ideal of a simplicial complex \( \Delta \), then the Stanley-Reisner ideal of its Alexander dual \( \Delta^\vee \) is \( I^\vee \).

Another related notion is componentwise linear ideals, introduced by Herzog and Hibi, to characterize sequentially Cohen-Macaulay ideals.
Let $I$ be a graded ideal of $R$ and let $I_{<d}$ be the ideal generated by all homogeneous polynomials of degree $d$ of $I$. A graded ideal $I$ of $R$ is called \textit{componentwise linear} if $I_{<d}$ has a linear resolution, for every $d$. Let $I$ be a square-free monomial ideal in a polynomial ring. The ideal generated by the square-free monomials of degree $d$ of $I$ is denoted by $I_{[d]}$. Herzog and Hibi in [9, Proposition 1.5] showed that the square-free ideal $I$ was componentwise linear if and only if $I_{[d]}$ had a linear resolution for every $d$.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $C \subseteq V(G)$ is a \textit{minimal vertex cover} of $G$ if: (1) every edge of $G$ is incident with one vertex in $C$, and (2) there is no proper subset of $C$ with the first property. In [8], Francisco and Van Tuyl showed that if $\mathcal{I}(G)$ was the ideal of a graph $G$, then

$$\mathcal{I}(G)_{[d]} = \{x_{i_1} \cdots x_{i_d} | \{x_{i_1}, \ldots, x_{i_d}\} \text{ is a vertex cover of } G \text{ of size } d\}.$$ 

In [9], Herzog and Hibi showed the following theorem to be used in the proof of Proposition 2.4.

\textbf{Theorem A.} Let $I$ be a square-free monomial ideal in a polynomial ring. Then $I^\vee$ is componentwise linear if and only if $R/I$ is sequentially Cohen-Macaulay.

Let $N(v)$ be the set of all adjacent vertices of $v$ and let $N[v] = N(v) \cup \{v\}$. Vertex decomposability was introduced by Provan and Billera [11] in the pure case, and extended to the non-pure case by Björner and Wachs [2]. We will use the following definition of vertex decomposable graphs which is an interpretation of the definition of vertex decomposable for the independence complex of a graph, as stated in [13, 16].

\textbf{Definition 2.1.} The independence complex of $G$ is vertex decomposable if $G$ is a totally disconnected graph (with no edges), or if

- $G \setminus v$ and $G \setminus N[v]$ are both vertex decomposable, and
- No independent set in $G \setminus N[v]$ is a maximal independent set in $G \setminus v$.

A vertex $v$ which satisfies in these conditions is called a shedding vertex.
The graph $G$ is called vertex decomposable if its independence complex is vertex decomposable. It is known that the any vertex decomposable graph is shellable and so is sequentially Cohen-Macaulay (see [16]).

For characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form $\theta_{n_1, \ldots, n_k}$, we have to distinguish among some cases, depending on $n_1, \ldots, n_k$, as follows.

**Proposition 2.2.** If $\{1, 2\} \subseteq \{n_1, \ldots, n_k\}$, then $\theta_{n_1, \ldots, n_k}$ is vertex decomposable and so is shellable and sequentially Cohen-Macaulay.

**Proof.** Two paths of length one and two form a triangle. Let $v, u$ and $w$ be its vertices such that $\deg(v) = 2$. The graphs $\theta_{n_1, \ldots, n_k} \setminus \{u\}$ and $\theta_{n_1, \ldots, n_k} \setminus N[u]$ are chordal and so they are vertex decomposable, by [16, Theorem 1]. For any independent set $F$ in $\theta_{n_1, \ldots, n_k} \setminus N[u]$, $F \cup \{v\}$ is an independent set in $\theta_{n_1, \ldots, n_k} \setminus \{u\}$. Therefore, $\theta_{n_1, \ldots, n_k}$ fulfills the conditions of Definition 2.1, which completes the proof. 

**Remark 2.3.** If in the above proposition, one assumes $\{n_1, \ldots, n_k\} = \{1, 2\}$, then the associated graph, $\theta_{n_1, \ldots, n_k}$, is chordal. These types of graphs are known to be vertex decomposable, by [16, Theorem 1].

A chordless path in a graph $G$ is a path $v_1, v_2, \ldots, v_k$ in $G$ with no edge $v_iv_j$ with $j \neq i + 1$. A simplicial $k$-path in $G$ is a chordless path $v_1, v_2, \ldots, v_k$ which cannot be extended on both endpoints to a chordless path $v_0, v_1, \ldots, v_{k-1}$ in $G$.

**Proposition 2.4.** Let $\{2, 3\} \subseteq \{n_1, \ldots, n_k\}$. Then, $\theta_{n_1, \ldots, n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.

**Proof.** Let $P_1 : u, x, v$ and $P_2 : u, y, z, v$ be two paths of length two and three in $\theta_{n_1, \ldots, n_k}$. Since the path $P : x, u, y$ is a simplicial 3-path, which is not a subgraph of any chordless $C_4$, by [16, Lemma 4.3] we deduce that $G$ is vertex decomposable.

**Proposition 2.5.** Let $\{n_1, \ldots, n_k\} = \{1, 4\}$. Then, $\theta_{n_1, \ldots, n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.
Proof. Each cycle other than $C_5$ in $\theta_{n_1, \ldots, n_k}$ has a chord and so, by [16, Theorem 1], it is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all sequentially Cohen-Macaulay graphs of the form $\theta_{n_1, \ldots, n_k}$, where $\{n_1, \ldots, n_k\} \neq \{2, 5\}$.

**Theorem 2.6.** Let $n_1, \ldots, n_k \neq \{2, 5\}$. Then, $\theta_{n_1, \ldots, n_k}$ is sequentially Cohen-Macaulay if and only if one of the following holds:

1. $\{1, 2\} \subseteq \{n_1, \ldots, n_k\}$.
2. $\{2, 3\} \subseteq \{n_1, \ldots, n_k\}$.
3. $\{1, 4\} = \{n_1, \ldots, n_k\}$.

Proof. “If”. Suppose that one of (1) to (3) holds. Then, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, the result holds.

“Only if”. Let $G = \theta_{n_1, \ldots, n_k}$ be a sequentially Cohen-Macaulay graph. The proof is by induction on $k$. If $k = 2$, then the graph is a cycle and so the result holds by [8, Proposition 4.1]. Let $k > 2$, $n_1 \leq \cdots \leq n_k$ and $P_i : x, x_{i,1}, \ldots, x_{i,n_i-1}, y$, for $1 \leq i \leq k$, be the paths which construct $G$. If $n_t \geq 6$, for some $t \geq 3$, then

$$H = G \setminus \bigcup_{i=t}^k (N[x_{i,2}] \cup N[x_{i,n_i-2}])$$

has a component of the form $\theta_{n_1, \ldots, n_{t-1}}$. So, by the induction hypothesis, (1) or (2) or (3) holds, for $\theta_{n_1, \ldots, n_{t-1}}$. If (1) or (2) holds for $\theta_{n_1, \ldots, n_{t-1}}$, then this holds, for $\theta_{n_1, \ldots, n_k}$. Let (3) holds for $\theta_{n_1, \ldots, n_{t-1}}$, but $\{n_1, \ldots, n_k\} \neq \{1, 4\}$. Let $S = \{j; n_j = 4\}$ and $H' = G \setminus \bigcup_{j \in S} N[x_{j,2}]$. Since $n_2 = 4$, then $H'$ has no path of length two, three and four. By the induction hypothesis, $H'$ is not sequentially Cohen-Macaulay, which is a contradiction by [14, Theorem 3.3].

So, we can assume that $n_k < 6$. Since $G$ has no vertex of degree one, it is not a bipartite graph by [14, Lemma 2.8]. Therefore, for $n_k = 2$, we have $n_1 = 1$ and so (1) holds. Similarly, if $n_k = 3$, then $n_i = 2$, for some $i$, and so (2) holds. If $n_k = 4$, then $G \setminus N[x_{k,2}]$ is $\theta_{n_1, \ldots, n_{k-1}}$. If (1), (2) or (3) holds, for $\theta_{n_1, \ldots, n_{k-1}}$, then the similar statement holds for $G$. So, assume that $n_k = 5$. Since $G$ is not bipartite, for some $i$ we have $n_i = 2$ or 4. If $n_i = 4$ for some $i$, then $H = G \setminus N[x_{i,2}]$ is sequentially Cohen-Macaulay and so (1) or (2) holds, which completes the result.
Otherwise, the assumption \( \{n_1, \ldots, n_k\} \neq \{2, 5\} \) shows that \( n_j = 1 \) or \( 3 \), for some \( j \), and so (1) or (2) holds.

Recently, Van Tuyl showed that in bipartite graphs, the three concepts vertex decomposability, shellability and sequentially Cohen-Macaulayness are equivalent; see [13, Theorem 2.10]. Using the proof of the above theorem, we have the same property for \( \theta_{n_1, \ldots, n_k} \), where \( \{n_1, \ldots, n_k\} \neq \{2, 5\} \).

**Theorem 2.7.** Let \( n_1, \ldots, n_k \neq \{2, 5\} \). Then, the followings are equivalent:

(i) \( \theta_{n_1, \ldots, n_k} \) is sequentially Cohen-Macaulay.

(ii) \( \theta_{n_1, \ldots, n_k} \) is shellable.

(iii) \( \theta_{n_1, \ldots, n_k} \) is vertex decomposable.

**Proof.** Note that (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i) always holds for any graph. It is enough to show that for these type of graphs, (i) \( \Rightarrow \) (iii). Let \( \theta_{n_1, \ldots, n_k} \) be a sequentially Cohen-Macaulay graph. Then, Theorem 2.6 shows that \( \theta_{n_1, \ldots, n_k} \) satisfies one of the relations of Theorem 2.6. Therefore, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, we deduce that \( \theta_{n_1, \ldots, n_k} \) is vertex decomposable. \( \square \)

In the following, we consider the case \( \{n_1, \ldots, n_k\} = \{2, 5\} \).

**Proposition 2.8.** Let \( \{n_1, \ldots, n_k\} = \{2, 5\} \). Then, \( \theta_{n_1, \ldots, n_k} \) is not vertex decomposable.

**Proof.** Let \( P_1, \ldots, P_s \) be the paths of length two in \( G = \theta_{n_1, \ldots, n_k} \) and \( P_{s+1}, \ldots, P_k \) be the paths of length five in \( G \). Consider the labeling for \( G \) such that \( P_j : u, \alpha_j, v \), for \( 1 \leq j \leq s \), and \( P_j : u, x_{j,1}, x_{j,2}, x_{j,3}, x_{j,4}, v \), for \( s + 1 \leq i \leq k \). We claim that no vertex of \( G \) is a shedding vertex to deduce that \( G \) is not vertex decomposable. For any \( s + 1 \leq j \leq k \), the independent set \( \{u, x_{s+1,4}, \ldots, x_{k,4}\} \) is maximal in both graphs \( G \setminus x_{j,2} \) and \( G \setminus N[x_{j,2}] \). For the other vertices of \( G \), the similar arguments hold. Therefore, \( G \) is not vertex decomposable. \( \square \)

Proposition 2.8 and Theorem 2.6 imply the following characterization of the vertex decomposable graphs of the form \( \theta_{n_1, \ldots, n_k} \).

**Theorem 2.9.** Let \( n_1, \ldots, n_k \) be a sequence of positive integers. Then, \( \theta_{n_1, \ldots, n_k} \) is vertex decomposable if and only if one of the followings holds:
(1) \( \{1, 2\} \subseteq \{n_1, \ldots, n_k\} \).
(2) \( \{2, 3\} \subseteq \{n_1, \ldots, n_k\} \).
(3) \( \{1, 4\} = \{n_1, \ldots, n_k\} \).

The next result extends Proposition 2.8 to show that for \( k = 3 \), those graphs are not even sequentially Cohen-Macaulay.

**Proposition 2.10.** The graphs \( \theta_{2,2,5} \) and \( \theta_{2,5,5} \) are not sequentially Cohen-Macaulay.

**Proof.** Consider the labeling for \( \theta_{2,2,5} \) and \( \theta_{2,5,5} \) as given in Figure 1 and Figure 2. By [8, Lemma 2.3], the minimal generators of \( I(\theta_{2,2,5})^\vee \), correspond to the minimal vertex covers of \( \theta_{2,2,5} \) and these minimal vertex covers correspond precisely to minimal prime ideals of \( I(\theta_{2,2,5}) \). Therefore, by finding the minimal prime ideals of \( I(\theta_{2,2,5}) \), the monomials
\[
x_1x_2x_4x_6, x_1x_3x_4x_6, x_2x_4x_6x_7x_8, x_1x_3x_5x_6, x_2x_4x_5x_7x_8, x_2x_3x_5x_7x_8, x_1x_3x_5x_7x_8,
\]
generate the ideal \( I(\theta_{2,2,5})^\vee \). With computation by CoCoA, we see that \( I(\theta_{2,2,5})^\vee \) has the minimal graded free resolution as
\[
0 \to R^3(-8) \to R^{12}(-7)(+)R(-8) \to R^{23}(-6) \to R^{14}(-5) \to R.
\]

Thus, it does not have a linear resolution. Therefore, \( \theta_{2,2,5} \) is not sequentially Cohen-Macaulay, by Theorem A.

Similarly, the minimal prime ideals of \( I(\theta_{2,5,5}) \) generate the ideal \( I(\theta_{2,5,5})^\vee \). By computation, we deduce that \( I(\theta_{2,5,5})^\vee \) has the minimal graded free resolution as:
\[
\cdots \to R^{55}(-10)(+)R(-11) \to R^{121}(-9) \to R^{124}(-8) \to R^{49}(-7) \to R.
\]

Thus, \( I(\theta_{2,5,5})^\vee \) does not have a linear resolution and so \( I(\theta_{2,5,5})^\vee \) is not componentwise linear. Therefore, \( \theta_{2,5,5} \) is not sequentially Cohen-Macaulay by Theorem A.

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**Figure 1**

**Figure 2**

\( \square \)
Sequentially Cohen-Macaulay graphs of form $\theta_{n_1, \ldots, n_k}$

In view of Proposition 2.8 and Proposition 2.10, we conjecture that the answer to the following questions is positive.

**Question 2.11.** Let $K > 2$ and \{n_1, \ldots, n_k\} = \{2, 5\}. Is $\theta_{n_1, \ldots, n_k}$ not shellable? Is $\theta_{n_1, \ldots, n_k}$ not sequentially Cohen-Macaulay?

Theorem 2.6 with [14, Theorem 2.9] enable us to get more examples of shellable and sequentially Cohen-Macaulay graphs.

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