GENERALIZED STEFFENSEN MEANS

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ABSTRACT. Using Steffensen’s inequality and log-convexity mean-value theorem, we introduce new means and then we establish their monotonicity properties. We also generalize some parts of theory given in [J. Jakšetić, J.E. Pečarić, Steffensens means, J. Math. Inequal. 2 (2008) 487-498].

1. Introduction and preliminary

The well-known Steffensen inequality reads as follows

**Theorem 1.1.** Suppose that $f$ is decreasing and $g$ is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then,

$$
\int_a^{b-\lambda} f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.
$$

(1.1)

The inequalities are reversed for an increasing function $f$.

In [2, p. 184] it is shown that condition $0 \leq g \leq 1$ in Theorem 1.1 can be replaced with a more general one as specified in the next theorem.
Theorem 1.2. Assume that $f$ and $g$ are integrable functions on $[a, b]$. Then, the inequalities in (1.1) hold for every decreasing function $f$ if and only if

\[
0 \leq \int_a^x g(t)dt \leq b - x \quad \text{and} \quad 0 \leq \int_x^b g(t)dt \leq x - a, \quad \text{for every} \quad x \in [a, b].
\] (1.2)

The following theorem represents a new generalization of mean-value theorem given in [1].

Theorem 1.3. Let $f \in C^1([a, b])$ be increasing and let $g$ be integrable function on $[a, b]$ such that (1.2) is valid and $\lambda = \int_a^b g(t)dt$. If $h \in C^1([f(a), f(b)])$, then there exist $\eta, \xi \in [f(a), f(b)]$ such that

\[
\begin{align*}
&\int_a^b h(f(t))g(t)dt - \int_a^a h(f(t))g(t)dt = h'(\xi) \left[ \int_a^b f(t)g(t)dt - \int_a^a f(t)dt \right] \\
&\text{and} \\
&\int_a^b h(f(t))g(t)dt - \int_a^b h(f(t))dt = h'(\eta) \left[ \int_a^b f(t)g(t)dt - \int_a^b f(t)dt \right].
\end{align*}
\] (1.3)

Proof. Since $h'$ is continuous on $[f(a), f(b)]$, there exist $m = \min h'$ and $M = \max h'$ both as real numbers. We first consider the function, $\tilde{h}(x) = Mx - h(x)$. Then, $h'(x) = M - h'(x) \geq 0$, $x \in [f(a), f(b)]$, and so $\tilde{h}$ is an increasing function. Applying Steffensen’s inequality, from Theorem 1.1 on increasing function $\tilde{h} \circ f$, we have

\[
0 \leq \int_a^b \tilde{h}(f(t))g(t)dt - \int_a^{a+\lambda} \tilde{h}(t)dt = M \int_a^b f(t)g(t)dt - \int_a^b h(f(t))g(t)dt + \\
-M \int_a^{a+\lambda} f(t)dt + \int_a^{a+\lambda} h(f(t))dt,
\]
that is,
\[
\int_{a}^{b} h(f(t))g(t)dt - \int_{a}^{a+\lambda} h(f(t))dt \leq M \left[ \int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} f(t)dt \right].
\]
Similarly, since \( \hat{h}(x) = h(x) - mx \) is an increasing function we can apply Steffensen’s inequality on increasing function \( \hat{h} \circ f \) and get
\[
m \left[ \int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} f(t)dt \right] \leq \int_{a}^{b} h(f(t))g(t)dt - \int_{a}^{a+\lambda} h(f(t))dt.
\]
We now conclude that there exists \( \xi \in [f(a), f(b)] \) satisfying (1.3). With the same technique one can prove existence of \( \eta \) in (1.4). □

**Remark 1.4.** It can be shown (see [1]) that if \( g \) is an integrable function that differs from the function \( x \mapsto 1_{[a, a+\lambda]}(x) \) on a set of positive measure, then the left hand side of (1.3) is different from 0. Similarly, if \( g \) is an integrable function that differs from the function \( x \mapsto 1_{[b-\lambda, b]}(x) \) on a set of positive measure, then the left hand side of (1.4) is different from 0.

**Corollary 1.5.** Let \( f \in C^1([a, b]) \) be a strictly monotone function and \( h_1, h_2 \in C^1([f(a), f(b)]) \), \( g \) integrable on \([a, b]\), with \( \lambda = \int_{a}^{b} g(t)dt \) and (1.5)
\[
0 \leq \int_{x}^{b} g(t)dt \leq b - x \text{ and } 0 \leq \int_{a}^{x} g(t)dt \leq x - a \quad \text{for every } x \in [a, b].
\]
Then, there exist \( \xi, \eta \in [f(a), f(b)] \) such that
\[
\int_{a}^{b} h_1(f(t))g(t)dt - \int_{a}^{a+\lambda} h_1(f(t))dt = \frac{h_1'(\xi)}{h_2'(\xi)},
\]
\[
\int_{a}^{b} h_2(f(t))g(t)dt - \int_{a}^{a+\lambda} h_2(f(t))dt = \frac{h_1'(\eta)}{h_2'(\eta)}.
\]

**Proof.** We show (1.6) first. Define the linear functional \( L(h) = \int_{a}^{b} h(f(t))g(t)dt - \int_{a}^{a+\lambda} h(f(t))dt \). Next, define \( \phi(t) = h_1(t)L(h_2) - h_2(t)L(h_1) \).
By Theorem 1.3, there exists \( \xi \in [f(a), f(b)] \) such that

\[
L(\phi) = \phi'(\xi) \left[ \int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} f(t)dt \right].
\]

From \( L(\phi) = 0 \), it follows that \( h_1'(\xi)L(h_2) - h_2'(\xi)L(h_1) = 0 \) and (1.6) is proved. The proof of (1.7) is quite similar. \( \square \)

In [1], we used Steffensen’s inequality to define Cauchy means of two numbers. In the next section, we will generalize this result by defining Cauchy means of general monotonic functions.

2. Main results

Let \( x, y \) be fixed, with \( 0 < x < y \). Let \( f \in C^1([x, y]) \) be strictly increasing function and \( h_1, h_2 \in C^1([f(x), f(y)]) \) monotonic functions, and \( g \) integrable function on \([x, y]\) such that \( \lambda = \int_{x}^{y} g(t)dt \) and \( g \) satisfies the corresponding condition (1.5).

Corollary 1.5 enables us to define various types of means, because if \( h_1'/h_2' \) has an inverse, then from (1.6) and (1.7) we have

\[
\xi = \left( \frac{h_1'}{h_2'} \right)^{-1} \left( \frac{y}{x} \int_{x}^{y} h_1(f(t))g(t)dt - \frac{x+\lambda}{x} \int_{x}^{x+\lambda} h_1(f(t))dt \right),
\]

and

\[
\eta = \left( \frac{h_1'}{h_2'} \right)^{-1} \left( \frac{y}{x} \int_{x}^{y} h_1(f(t))g(t)dt - \frac{y-\lambda}{y} \int_{y-\lambda}^{y} h_1(f(t))dt \right),
\]

which means that \( \xi \) and \( \eta \) are means of numbers \( x \) and \( y \), for given functions \( f \) and \( g \). Specially, if we take substitutions \( h_1(t) = t^r, h_2(t) = t^s \) in (2.1) and (2.2), we obtain the following expressions,
Generalized Steffensen means

\[ S_1(f,g;x,y;r,s) = \left[ s \left( \int_x^y f^r(t)g(t)dt - \int_x^y f^r(t)dt \right) \right] \frac{1}{r-s}, \]

and

\[ S_2(f,g;x,y;r,s) = \left[ s \left( \int_x^y f^s(t)g(t)dt - \int_x^y f^s(t)dt \right) \right] \frac{1}{r-s}, \]

where, \((r - s)r \cdot s \neq 0\).

Continuous extensions of (2.3) are:

\[ S_1(f,g;x,y;0,s) = S_1(f,g;x,y;0,s) = \]

\[ \left[ \frac{\int_x^y f^s(t)g(t)dt - \int_x^y f^s(t)dt}{s \left( \int_x^y g(t) \ln f(t)dt - \int_x^y \ln f(t)dt \right)} \right] \frac{1}{s}, s \neq 0 \]

\[ S_1(f,g;x,y;s,s) = \exp \left( \frac{\int_x^y g(t) \ln f(t)dt - \int_x^y \ln f(t)dt - \frac{1}{s} \int_x^y f^r(t)dt - \frac{1}{s^2} \int_x^y f^r(t)dt}{s \left( \int_x^y f^s(t)g(t)dt - \int_x^y f^s(t)dt \right)} \right), s \neq 0 \]

Continuous extensions of (2.4) are:
Now, we establish monotonicity properties of the new means.

**Theorem 2.1.** Let \( r \leq u, \ s \leq v \). Then,

\[
S_1(f, g; x, y; r, s) \leq S_1(f, g; x, y; u, v)
\]

and

\[
S_2(f, g; x, y; r, s) \leq S_2(f, g; x, y; u, v).
\]

For the proof, we need the following two lemmas.

**Lemma 2.2.** Let \( f \) be a log-convex function. If \( x_1 \leq y_1, \ x_2 \leq y_2, \ x_1 \neq x_2, \ y_1 \neq y_2 \), then the following inequality holds:

\[
(2.5) \quad \left( \frac{f(x_2)}{f(x_1)} \right)^{1/(x_2-x_1)} \leq \left( \frac{f(y_2)}{f(y_1)} \right)^{1/(y_2-y_1)}.
\]

**Proof.** This follows from Remark 1.2. in [2], \( \square \)

Observe that

\[
S_1(f, g; x, y; r, s) = \left( \frac{\phi(r)}{\phi(s)} \right)^{1/s},
\]
where,

\[
\phi(r) = \begin{cases} 
\frac{1}{r} \left( \int \frac{y}{x} f'(t)g(t)dt - \int \frac{x^+}{x} f'(t)dt \right), & r \neq 0, \\
\int \frac{y}{x} g(t) \ln f(t)dt - \int \frac{x^+}{x} \ln f(t)dt, & r = 0.
\end{cases}
\]

Similarly,

\[
S_2(f; g; x, y; r, s) = \left( \frac{\psi(r)}{\psi(s)} \right)^{\frac{1}{r-s}},
\]

where,

\[
\psi(r) = \begin{cases} 
\frac{1}{r} \left( \int \frac{y}{x} f'(t)dt - \int \frac{x^+}{x} (t)dt \right), & r \neq 0, \\
\int \ln f(t)dt - \int \ln g(t)dt, & r = 0.
\end{cases}
\]

Let us observe that \( \lim_{r \to 0} \phi(r) = \phi(0) \) and \( \lim_{r \to 0} \psi(r) = \psi(0) \), meaning that \( \phi \) and \( \psi \) are continuous functions.

\[\square\]

**Lemma 2.3.** The functions \( \phi \) and \( \psi \) defined by (2.6) and (2.7) are log-convex functions.

**Proof.** Consider the following function,

\[
h(x) = p^2 \varphi_r(x) + 2pq \varphi_z(x) + q^2 \varphi_s(x)
\]
where \( z = \frac{r+s}{2} \) and \( p, q \in \mathbb{R} \),

and

\[
\varphi_u(x) = \begin{cases} 
x^u, & u \neq 0, \\
\ln x, & u = 0.
\end{cases}
\]

Now,

\[
h'(x) = p^2 x^{r-1} + 2pq x^{z-1} + q^2 x^{s-1}
= \left(p x^{(r-1)/2} + q x^{(s-1)/2}\right)^2 \geq 0.
\]

This implies that \( h \) is monotonically increasing. Since \( f \) is an increasing function, then \( h \circ f \) is an increasing function. Then, the following Steffensen’s inequalities from Theorem 1.2 are satisfied:
\[ \int_{x}^{y} h(f(t))g(t)\,dt - \int_{x}^{x+\lambda} h(f(t))\,dt \geq 0 \]

and
\[ \int_{y-\lambda}^{y} h(f(t))\,dt - \int_{x}^{y} h(f(t))g(t)\,dt \geq 0. \]

From (2.8) it then follows:
\[ p^2\phi(r) + 2pq\phi(z) + q^2\phi(s) \geq 0, \quad \text{where} \quad z = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R}. \]

This implies:
\[ \phi^2\left(\frac{r+s}{2}\right) \leq \phi(r)\phi(s), \]
that is, \( \phi \) is a log-convex function in the Jensen sense. Since we have shown that \( \phi \) is a continuous function, we conclude that \( \phi \) is log-convex function. The log-convexity of \( \psi \) can be deduced in a similar way. \( \square \)

**Proof.** [Proof of Theorem 2.1] We now apply inequality (2.5) from Lemma 2.2 for \( f = \phi, r \leq u, s \leq v, r \neq s, u \neq v \) \( (r, t, u, v \neq 0) \) to deduce:
\[ \left( \frac{\phi(r)}{\phi(s)} \right)^{\frac{1}{r-s}} \leq \left( \frac{\phi(u)}{\phi(v)} \right)^{\frac{1}{u-v}}. \]

Since \( (r, s) \mapsto S_1(f, g; x, y; r, s) \) is continuous, we have, for \( r \leq u, s \leq v, \)
\[ S_1(f, g; x, y; r, s) \leq S_1(f, g; x, y; u, v). \]

The same arguments stand for \( S_2(f, g; x, y; r, s) \). \( \square \)

**REFERENCES**


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