AUTOMATIC CONTINUITY OF HIGHER DERIVATIONS ON $JB^*$-ALGEBRAS

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Abstract. In this paper we study higher derivations from $JB^*$-algebras into Banach Jordan algebras. We show that every higher derivation $\{d_m\}$ from a $JB^*$-algebra $A$ into a $JB^*$-algebra $B$ is continuous provided that $d_0$ is a $*$-homomorphism. Also it is proved that every Jordan higher derivation from a commutative $C^*$-algebra or from a $C^*$-algebra which has minimal idempotents and is the closure of its socle is continuous.

1. Introduction

Let $A$ and $B$ be algebras (associative or non-associative). By a higher derivation of rank $k$ ($k$ might be $\infty$) we mean a family of linear mappings $\{d_m\}_{m=0}^k$ from $A$ into $B$ such that

$$d_m(ab) = \sum_{j=0}^{m} d_j(a)d_{m-j}(b), \quad (a, b \in A, \quad m = 0, 1, 2, \ldots, k).$$

It is clear that $d_0$ is a homomorphism. Higher derivations were introduced by Hasse and Schmidt [8], and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find some of algebraic results concerning these mappings in [1, 4, 6, 14, 16, 17]. They are also studied in other contexts. In [19] higher derivations are applied to study...
generic solving of higher differential equations.

A standard example of a higher derivation of rank $k$ is the family $\{\frac{D^{m}}{m!}\}_{m=0}^{k}$, where $D$ is an ordinary derivation of an algebra $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are normed algebras then a higher derivation $\{d_{m}\}$ is said to be continuous, whenever every $d_{m}$ is continuous. It is known that every derivation on a semisimple Banach algebra is continuous [13]. Ringrose [15] proved that every derivation from a $C^{\ast}$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module is continuous. In [9] derivations from $JB^{\ast}$-algebras into Banach Jordan modules were studied and continuity of these mappings were proved in certain cases. Loy in [12] proved that if $\mathcal{A}$ is an $(F)$-algebra which is a subalgebra of a Banach algebra $\mathcal{B}$ of power series, then every higher derivation $\{d_{m}\} : \mathcal{A} \to \mathcal{B}$ is automatically continuous. Jewell [11], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_{0}) \subseteq \ker(d_{m})$, for all $m \geq 1$. Villena [20] proved that every higher derivation from a unital Banach algebra $\mathcal{A}$ into $\mathcal{A}/\mathcal{P}$, where $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ with infinite codimension, is continuous. Also the range problem of continuous higher derivations was studied in [14].

In this paper we study automatic continuity of higher derivations from $JB^{\ast}$-algebras. Section 2 is devoted to some concepts which are needed in the sequel. In Section 3 we prove that a higher derivation from a $JB^{\ast}$-algebra into another $JB^{\ast}$-algebra is continuous provided that $d_{0}$ is a $\ast$-homomorphism. Also we will show that every (Jordan) higher derivation from a commutative $C^{\ast}$-algebra or from a $C^{\ast}$-algebra which has minimal idempotents and is the closure of its socle (e. g. $K(\mathcal{H})$) into a Banach Jordan algebra is continuous. These are in fact generalizations of some results in [9].

2. Preliminaries

Let $\mathcal{A}$ be a Jordan algebra and let $\mathcal{X}$ be a vector space over the same field as $\mathcal{A}$. Then $\mathcal{X}$ is said to be a Jordan $\mathcal{A}$-module if there is a pair of bilinear mappings (called module operations), $(a, x) \mapsto a.x$, $(a, x) \mapsto x.a$, from $\mathcal{A} \times \mathcal{X} \to \mathcal{X}$ such that for all $a, b \in \mathcal{A}$ and all $x \in \mathcal{X}$ the following conditions hold:

(i) $a.x = x.a$;
(ii) $a.(a^{2}.x) = a^{2}.(a.x)$;
(iii) $2((x.a).b).a + x.(a^{2}.b) = 2(x.a)(a.b) + (x.b).a^{2}$.
A linear subspace $S$ of $X$ is called a submodule if

$$\mathcal{A}S := \{a.x: a \in \mathcal{A}, x \in S\} \subseteq S.$$ 

If $\mathcal{A}$ is a Banach Jordan algebra and $X$ is a Banach space which is a Jordan $\mathcal{A}$-module then $X$ is said to be a weak Jordan $\mathcal{A}$-module whenever the mapping $x \mapsto a.x$, from $X \rightarrow X$ is continuous, for all $a \in \mathcal{A}$; and $X$ is called a Banach Jordan $\mathcal{A}$-module if the mapping $(a, x) \mapsto a.x$, from $A \times X \rightarrow X$ is continuous, or equivalently, if there exists $M > 0$ such that $\|a.x\| \leq M \|a\| \|x\|$ ($a \in \mathcal{A}, x \in X$).

**Example 2.1.** (i) Every Banach Jordan algebra $\mathcal{A}$ is a Banach Jordan $\mathcal{A}$-module whenever we consider its own product as the module operation.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are Jordan algebras and $\theta: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism, then $\mathcal{B}$ can be considered as a Jordan $\mathcal{A}$-module with module operation

$$a.b = \theta(a)b \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

In this case we will say that $\mathcal{B}$ is an $\mathcal{A}$-module via the homomorphism $\theta$. If $\mathcal{A}$ and $\mathcal{B}$ are Banach Jordan algebras then it is easy to see that $\mathcal{B}$ is a weak Jordan $\mathcal{A}$-module.

(iii) The topological dual $\mathcal{A}^*$ of $\mathcal{A}$, with module operation $(a, f) \mapsto a.f$ defined by

$$(a.f)(b) = f(ab) \quad (a, b \in \mathcal{A}, f \in \mathcal{A}^*),$$

is a Banach Jordan $\mathcal{A}$-module.

(iv) If $\mathcal{A}$ is a Banach algebra and $X$ is a Banach (respectively weak) $\mathcal{A}$-module, then we may consider $\mathcal{A}$ as a Jordan algebra with Jordan product $(a, b) \mapsto \frac{ab + ba}{2}, \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$. Then $X$ with the module operation $a.x = \frac{ax + xa}{2}$, is a Banach (respectively weak) Jordan $\mathcal{A}$-module. Here the mappings $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$, $\mathcal{A} \times X \rightarrow X$, denote the associative module operations of $\mathcal{A}$ on $X$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Jordan $\mathcal{A}$-modules. Then a linear mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be a module homomorphism if $T(a.x) = a.T(x)$ ($a \in \mathcal{A}, x \in \mathcal{X}$). In Example 2.1 (ii), $\theta$ is a module homomorphism.

Let $\mathcal{A}$ be a Jordan algebra and let $\mathcal{X}$ be a Jordan $\mathcal{A}$-module. Then $\mathcal{A} \oplus \mathcal{X}$ with product $(a_1 + x_1)(a_2 + x_2) = a_1a_2 + a_1x_2 + a_2x_1$, is a Jordan algebra which is called the split null extension of $\mathcal{A}$ and $\mathcal{X}$. In fact a linear space $\mathcal{X}$ is a Jordan $\mathcal{A}$-module if and only if this split null extension is a Jordan algebra [10].
Corresponding to \((a, 0) \in \mathcal{A} \oplus \mathcal{X}\) with \(a \in \mathcal{A}\), as in any Jordan algebra, we define the linear operators \(R_a\) and \(U_a\) on \(\mathcal{A} \oplus \mathcal{X}\) as follows

\[ R_a(u) = au, \quad U_a(u) = 2a(u) - a^2 u \quad (u \in \mathcal{A} \oplus \mathcal{X}). \]

We feel free to use the notation \(R_a\) and \(U_a\) for the same operators on \(\mathcal{A}\).

For every \(x, y\) in a Jordan algebra, set \([R_x, R_y] := R_x R_y - R_y R_x\). We recall that each \(x, y, z\) in a Jordan algebra satisfy

\[ [R_{xy}, R_z] + [R_{xz}, R_y] + [R_{yz}, R_x] = 0, \tag{2.1} \]

which is the identity (O1) in Section 1.7 of [10]. For a submodules of \(\mathcal{X}\), set

\[
\mathcal{R}(S) := \{ a \in \mathcal{A}: R_a(x) = 0 \text{ for all } x \in S \}, \\
\mathcal{Q}(S) := \{ a \in \mathcal{A}: U_a(x) = 0 \text{ for all } x \in S \}, \\
\mathcal{I}(S) := \{ a \in \mathcal{R}(S): ab \in \mathcal{R}(S) \text{ for all } b \in \mathcal{A} \}.
\]

Note that if \(S\) is a submodule, then it is an ideal of \(\mathcal{A} \oplus \mathcal{X}\), and \(\mathcal{I}(S)\) is actually \(\text{ann}(S)\) in view of Zelmanov, which is an ideal by Lemma 3(b) of [21]. Here we give the proof for the sake of convenience.

**Lemma 2.2.** Let \(\mathcal{A}\) be a Jordan algebra and let \(\mathcal{X}\) be a Jordan \(\mathcal{A}\)-module. If \(S\) is a submodule then

(i) \(\mathcal{I}(S)\) is the largest ideal of \(\mathcal{A}\) contained in \(\mathcal{R}(S)\);

(ii) \(\mathcal{R}(S) \cap \mathcal{Q}(S) = \{ a \in \mathcal{A}: a^2 \in \mathcal{R}(S) \}\);

(iii) \(\mathcal{I}(S) \subseteq \mathcal{R}(S) \cap \mathcal{Q}(S)\).

**Proof.** (i) It is easy to see that each ideal of \(\mathcal{A}\) contained in \(\mathcal{R}(S)\) is a subset of \(\mathcal{I}(S)\). We show that \(\mathcal{I}(S)\) is an ideal. Suppose that \(a \in \mathcal{I}(S)\) and \(b \in \mathcal{A}\). Then by definition of \(\mathcal{I}(S)\), \(ab \in \mathcal{R}(S)\). Now to see that \(ab \in \mathcal{I}(S)\) it is enough to show that \((ab)c \in \mathcal{R}(S)\), for each \(c \in \mathcal{A}\). We consider (2.1) in the Jordan algebra \(\mathcal{A} \oplus \mathcal{X}\), for \(x = a, y = b, z \in \mathcal{S}\) and take \(c \in \mathcal{A}\). Since \(S\) is a submodule and \(a \in \mathcal{I}(S)\), it follows that \(R_z R_{ab}(c) = 0\), or equivalently, \((ab)c \in \mathcal{R}(S)\). Parts (ii) and (iii) are easily verified. \(\square\)

3. Automatic continuity of Higher derivations from \(JB^*-\)algebras

First of all we recall that a real Banach Jordan algebra \(\mathcal{A}\) is a \(JB\)-algebra whenever \(\|a^2\| = \|a\|^2\) and \(\|a^2\| \leq \|a^2 + b^2\|\), for all \(a, b \in \mathcal{A}\).
$A$. A complex Banach Jordan algebra $A$ is said to be a $JB^*$-algebra whenever there is an algebra involution $*$ on $A$ such that $\|a^*\| = \|a\| \text{ and } \|U_a(a^*)\| = \|a\|^2$, for all $a \in A$. For a subset $C$ of a $JB^*$-algebra $A$, set $C_h := \{a \in C: a = a^*\}$. Then $A_h$ is a $JB$-algebra and $A = A_h + iA_h$. If $a \in A_h$ then $C^*(a)$, the $JB^*$-subalgebra of $A$ generated by $a$ (or by $a, 1$ if $A$ is unital), is a $C^*$-algebra. Clearly each $C^*$-algebra with respect to its Jordan product is a $JB^*$-algebra. The reader is referred to [7] for more details on $JB$-algebras and $JB^*$-algebras. From now on throughout this section we assume that $A$ is a unital $JB^*$-algebra, $B$ is a Banach Jordan algebra and $\{d_m\}$ is a higher derivation of infinite rank from $A$ into $B$ with continuous $d_0$. For each $m = 0, 1, 2, \ldots$, set

$$S_m := \{b \in B : \exists \{a_n\} \subseteq A \text{ s.t. } a_n \to 0 \text{ and } d_m(a_n) \to b\},$$

which is called the separating space of $d_m$. This is a closed linear subspace of $B$ ([5], Theorem 5.1.2) and by the closed graph theorem $d_m$ is continuous if and only if $S_m = \{0\}$. Therefore $\{d_m\}$ is continuous if and only if $S_m = \{0\}$, for all $m \geq 0$. If we consider $B$ as a Jordan $A$-module via the homomorphism $d_0$ as in Example 2.1 (ii), then $d_1$ would be a derivation from $A$ into $B$. With the assumption on $d_0$ we have $S_0 = \{0\}$ and it is easy to see that $S_1$ is a submodule of $B$. In general $S_m$ is not a submodule for $m \geq 2$, but if $d_0, d_1, \ldots d_{m-1}$ are assumed to be continuous, then $d_m$ would be an intertwining map and hence $S_m$ is a submodule. Using the same notations as in Section 2, set $R_m := R(S_m)$, $Q_m := Q(S_m)$ and $T_m := T(S_m)$. If $d_0, \ldots, d_{m-1}$ are continuous then we have

$$R_m = \{a \in A : R_a d_m \text{ is continuous } \} = \{a \in A : d_m R_a \text{ is continuous} \},$$

and

$$Q_m = \{a \in A : U_a d_m \text{ is continuous } \} = \{a \in A : d_m U_a \text{ is continuous} \}.$$

Before we prove the next lemma, we recall that a subalgebra $C$ of a Jordan algebra $A$ is said to be strongly associative if $[R_a, R_b] = 0$, for all $a, b \in C$. By Example 1.8.1 of [10], for each $a \in A$, the subalgebra of $A$ generated by $a$, (or by $a, 1$ if $A$ is unital) is strongly associative and by ([10] Lemma 1.8.8), if $a, b$ lie in a strongly associative subalgebra, then $U_{ab} = U_a U_b$.

**Lemma 3.1.** Let $A$ be a $JB^*$-algebra. Suppose that $X$ is a Banach Jordan $A$-module, $Y$ is a weak Jordan $A$-module and $T : X \to Y$ is
a module homomorphism. If \( a \in A \), and \( \{f_n\} \subseteq C^*(a) \) is such that 
\( f_if_j = 0 \) \( (i \neq j) \), then \( U_{f_n}T \) is continuous for all but a finite number of 
\( n \)'s.

**Proof.** Suppose that \( U_{f_n}T \) is discontinuous for infinitely many \( n \)'s. By 
considering a subsequence we may assume that \( U_{f_n}T \) is discontinuous 
for each \( n \). Let \( M_n \) and \( K_n \) be the norms of the bounded linear operators 
\( x \mapsto U_{f_n}(x), \mathcal{X} \longrightarrow \mathcal{X} \), and \( y \mapsto U_{f_n}(y), \mathcal{Y} \longrightarrow \mathcal{Y} \), respectively. Note 
that \( M_n, K_n > 0 \) for each \( n \); otherwise \( U_{f_n}T = T U_{f_n} = 0 \) which is 
continuous. Choose a sequence \( \{x_n\} \) in \( \mathcal{X} \) such that 
\[
\|x_n\| \leq 2^{-n}/M_n, \\
\|U_{f_n}T(x_n)\| \geq nK_n.
\]
Take \( z = \sum_{n=1}^{\infty} U_{f_n}(x_n) \). By strong associativity of \( C^*(a) \) as a subalgebra 
of \( A \oplus \mathcal{X} \) and \( A \oplus \mathcal{Y} \), we have \( U_{f_i}U_{f_j} = U_{f_if_j} = 0 \) \( (i \neq j) \), on 
\( A \oplus \mathcal{X} \) and \( A \oplus \mathcal{Y} \). Since \( T \) is a module homomorphism, \( K_n\|T(z)\| \geq 
\|U_{f_n}T(z)\| = \|TU_{f_n}(z)\| = \|T(U_{f_n}^2(x_n))\| = \|U_{f_n}^2(Tx_n)\| \geq nK_n. 
Therefore \( \|T(z)\| \geq n \) for each \( n \), which is impossible. So the result 
holds. \( \square \)

**Remark 3.2.** Suppose that \( B \) is a Jordan algebra. Then \( B_m := \bigoplus_{m+1}^{m} B \) is a Jordan algebra with the product defined by 
\[
(x_0, x_1, \ldots, x_m)(y_0, y_1, \ldots, y_m) = (x_0y_0, x_0y_1 + x_1y_0, \ldots, \sum_{i=0}^{m} x_iy_{m-i}),
\]
for all \( (x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m) \in B_m. \) Clearly, this product is 
commutative. Suppose that \( \bar{x} = (x_0, x_1, \ldots, x_m), \bar{y} = (y_0, y_1, \ldots, y_m) \in B_m. \) Then the \( k^{th} \) entries of \( \bar{x}(\bar{x}^2\bar{y}) \) and \( \bar{x}^2(\bar{x}\bar{y}) \) are
\[
\sum_{l=0}^{k} x_l(\sum_{j=0}^{k-l} x_j x_{j-1}) y_{k-j-l}, \tag{3.1}
\]
and
\[
\sum_{l=0}^{k-1} x_l x_{l-1}(\sum_{j=0}^{k-l} x_j y_{k-j-1}), \tag{3.2}
\]
respectively. By identities \((O2)\) and \((O3)\) in Section 1.7 of [10], (3.1) and 
(3.2) are equal, and hence \( B_m \) is a Jordan algebra. Furthermore, let \( B \) be
a Banach Jordan algebra. Define a norm on $B_m$ by $\| (x_0, x_1, \ldots, x_m) \|_0 = \sum_{i=0}^{m} \| x_i \|$. Then $\| \cdot \|_0$ is a complete norm on $B_m$ and it is easy to see that

$$\|(x_0, x_1, \ldots, x_m)(y_0, y_1, \ldots, y_m)\|_0 \leq \|(x_0, x_1, \ldots, x_m)\|_0 \|(y_0, y_1, \ldots, y_m)\|_0,$$

for all $(x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m) \in B_m$. Therefore $B_m$ is a Banach Jordan algebra.

Lemma 3.3. Suppose that $A$ is a $JB^*$-algebra and $B$ is a Banach Jordan algebra. Let $\{d_m\} : A \rightarrow B$ be a higher derivation with continuous $d_0$. Let $a \in A_h$ and let $\{f_n\} \subseteq C^*(a)$ be such that $f_if_j = 0$ ($i \neq j$). Then for each $m = 0, 1, 2, \ldots$, we have $f_n^2 \in Q_m$, for all but a finite number of $n$’s.

Proof. Consider a fixed $m$, and let $B_m$ be as in Remark 3.2. We define $\theta_m : A \rightarrow B_m$, $a \mapsto (d_0(a), d_1(a), \ldots, d_m(a))$.

Then $\theta_m$ is a homomorphism and $B_m$ is a weak Jordan $A$-module via the homomorphism $\theta_m$. Also as in Example 2.1 (ii), $\theta_m$ is a module homomorphism. We have $U_fU_{f_i} = U_{f_i}U_f = 0$ ($i \neq j$), on the split null extension of $A$ and $B_m$. Hence by Lemma 3.1, $U_f^2\theta_m$ is continuous for all but a finite number of $n$’s. Thus for such $n$’s, $U_f^2d_1, \ldots, U_f^2d_m$ are continuous and it follows that $f_n^2 \in Q_m$, for all but a finite number of $n$’s. □

Theorem 3.4. Let $A$ be a $JB^*$-algebra and let $B$ be a Jordan Banach algebra. Suppose that $\{d_m\}$ is a higher derivation from $A$ into $B$ with continuous $d_0$. Then the following assertions hold.

(i) If $a \in A_h$ and $\Delta$ is the maximal ideal space of $C^*(a)$, then for every $m = 1, 2, \ldots$, the set $F_m = \{ \lambda \in \Delta : \lambda(Q_m \cap C^*(a)) = \{0\} \}$ is finite.

(ii) If $I$ is a closed ideal of $A$ containing $Q_m$, then every element in the $JB$-algebra $(A\Delta^m)_h$ has finite spectrum.

(iii) If $d_1, \ldots, d_{m-1}$ are continuous and $K$ is a closed ideal of $A$ contained in $Q_m$, then $d_m|_K$ is continuous.

(iv) If $d_1, \ldots, d_{m-1}$ are continuous and $L$ is an ideal of $A$ such that $d_m|_L$ is continuous, then $L \subseteq I_m \subseteq Q_m$. 
Proof. (i) If $F_n$ is infinite, then we may find an infinite sequence \( \{\lambda_k\} \subseteq \Delta \) and a sequence \( \{V_k\} \) of open subsets of \( \Delta \) such that \( V_j \cap V_k = \emptyset \) \((j \neq k)\), and \( \lambda_k \in V_k \), for each \( k \). For every \( k \in \mathbb{N} \), choose \( f_k \in C^*(a) \) such that \( f_k(\lambda_k) \neq 0 \) and \( f_k(\Delta \backslash V_k) = \{0\} \). Then \( f_k f_j = 0 \) \((k \neq j)\), and \( f_k^2 \notin \mathcal{Q}_m \) which contradicts Lemma 3.3.

(ii) Let \( I \) be a closed ideal in \( \mathcal{A} \) such that \( \mathcal{Q}_m \subseteq I \), for all \( m = 0, 1, 2, \ldots \).

For each \( a \in \mathcal{A}_h \), we have
\[
\{ \lambda \in \Delta : \lambda(\mathcal{I} \cap C^*(a)) = \{0\} \} \subseteq \{ \lambda \in \Delta : \lambda(\mathcal{Q}_m \cap C^*(a)) = \{0\} \}.
\]
Hence by (i) the left hand side is a finite set and as in Theorem 12.2 of [18], \( \frac{C^*(a)}{\mathcal{C}^*(a) / \mathcal{I}} \) is finite dimensional, and since the closed *-subalgebra of \( \mathcal{A} / \mathcal{I} \) generated by \( a \) and \( 1 \) is isomorphic to \( \frac{C^*(a)}{\mathcal{C}^*(a) / \mathcal{I}} \), the result holds.

(iii) We show that \( d_m \) is bounded on bounded subsets of \( \mathcal{K}_h \). On the contrary suppose that there is a sequence \( \{a_n\} \subseteq \mathcal{K}_h \) such that \( a_n \to 0 \) and \( \| d_m(a_n) \| \to \infty \). We may assume that \( \sum_{n=1}^{\infty} \| a_n \|^2 \leq 1 \). Let \( b = (\sum_{n=1}^{\infty} a_n^2)^{1/8} \). Then \( b \geq 0 \), \( \| b \| \leq 1 \) and \( a_n^2 \leq b^8 \) \((n \in \mathbb{N})\).

By [9] Lemma 1.7, for each \( n \in \mathbb{N} \) there exists \( u_n \in \mathcal{K}_h \) such that \( \| u_n \| \leq 2 \), \( \| b^{1/4} \| \leq 2 \) and \( a_n = U_b(u_n) \). Hence \( d_m(a_n) = d_m U_b(u_n) \). Since \( \mathcal{K} \subseteq \mathcal{Q}_m \), we have \( b \in \mathcal{Q}_m \) and so \( d_m U_b \) is continuous. Now it follows that \( \| d_m(a_n) \| \leq \| d_m U_b \| \| u_n \| \leq 2 \| d_m U_b \| \), which is a contradiction.

(iv) Suppose that \( d_m \mid \mathcal{L} \) is continuous. Take \( a \in \mathcal{S}_m \). Then there is a sequence \( \{a_n\} \subseteq \mathcal{A} \) such that \( a_n \to 0 \) and \( d_m(a_n) \to a \). Let \( b \in \mathcal{L} \). Since \( d_1, \ldots, d_{m-1} \) are continuous it follows that
\[
d_m(ba_n) = d_0(b)d_m(a_n) + d_1(b)d_{m-1}(a_n) + \ldots + d_m(b)d_0(a_n) \to ba.
\]
Since \( ba_n \in \mathcal{L} \) and \( d_m \mid \mathcal{L} \) is continuous, \( ba = 0 \). This means that \( b \in \mathcal{R}_m \) and hence \( \mathcal{L} \subseteq \mathcal{R}_m \). But \( \mathcal{I}_m \) is the largest ideal of \( \mathcal{A} \) contained in \( \mathcal{R}_m \), so we have \( \mathcal{L} \subseteq \mathcal{I}_m \subseteq \mathcal{Q}_m \).

Corollary 3.5. Let \( \mathcal{A} \) be a JB*-algebra and let \( \mathcal{B} \) be a Banach Jordan algebra. Suppose that \( \{d_m\} \) is a higher derivation from \( \mathcal{A} \) into \( \mathcal{B} \) with continuous \( d_0 \). If \( \mathcal{K} \) is a closed ideal of \( \mathcal{A} \) contained in \( \bigcap \mathcal{Q}_m \), then \( d_m \mid \mathcal{K} \) is continuous for all \( m \).

Proof. Similar to the proof of Theorem 3.4 (iii).

Theorem 3.6. Let \( \{d_m\} \) be a higher derivation of a JB*-algebra \( \mathcal{A} \) into a Banach Jordan algebra \( \mathcal{B} \) such that \( d_0 \) is continuous. Then \( \{d_m\} \) is
continuous if and only if \((Q_m)_h := \{a \in Q_m: \ a = a^*\}\) is a real linear subspace of \(A_h\), for all \(m \in \mathbb{N}\).

**Proof.** If \(\{d_m\}\) is continuous then \(Q = A\), and so \((Q_m)_h\) is real linear. Conversely let \((Q_m)_h\) be real linear. Since \(d_1\) is a derivation, by ([9] Theorem 2.2), \(d_1\) is continuous. Suppose by induction that each \(d_i\) \((i < m)\) is continuous. Then \(S_m\) is a submodule of \(B\), and \(U_{A_h}(Q_m)_h \subseteq (Q_m)_h\), hence \((Q_m)_h\) is an ideal of \(A_h\). By Theorem 3.4 (iii), \(d_m\) is continuous on \((Q_m)_h + i(Q_m)_h\). Hence \((Q_m)_h + i(Q_m)_h \subseteq I_m \subseteq Q_m\) and so \(I_m = (Q_m)_h + i(Q_m)_h\). Let \(\pi: A \to \frac{A}{I_m}\) be the canonical quotient map. By Theorem 3.4 (ii) every element in \((\frac{A}{I_m})_h\) has finite spectrum. But \((\frac{A}{I_m})_h = \frac{A_h}{(I_m)_h}\) is a semisimple real Banach Jordan algebra in which every element has non-empty finite spectrum and by [2] it is reduced, that is, there exist idempotents \(\pi(e_1), \ldots, \pi(e_n) \in (\frac{A}{I_m})_h\) such that \(\pi(e_i)\pi(e_j) = 0, (i \neq j)\), \(\sum_{i=1}^n \pi(e_i) = 1\), and \(U_{\pi(e_i)}(\frac{A}{I_m})_h = \mathbb{R}\pi(e_i), (i = 1, \ldots, n)\). Since each \(\pi(e_i)\) is self-adjoint, \(\pi(e_i^*e_i) = \pi(e_i)\), \(i = 1, \ldots, n\), and so \(\pi(e_i^*e_i), \ldots, \pi(e_n^*e_n)\) are idempotents in \((\frac{A}{I_m})_h\) with sum 1 such that \(\pi(e_i^*e_i)\pi(e_j^*e_j) = 0, (i \neq j)\). Hence by replacing \(e_i\) with \(e_i^*e_i\), if necessary, we may assume that each \(e_i\) is self-adjoint. Suppose that \(\{a_k\} \subseteq A_h\) and \(a_k \to 0\). Then \(\pi(a_k) \to 0\), and for each \(i = 1, \ldots, n\), and each \(k \in \mathbb{N}\), there exists \(\lambda_{ik} \in \mathbb{R}\) such that

\[
U_{\pi(e_i)}(\pi(a_k)) = \lambda_{ik} \pi(e_i).
\]

Hence \(\lambda_{ik} \pi(e_i) \to 0\) as \(k \to \infty\), and so \(\lambda_{ik} \to 0\) as \(k \to \infty\). By (3.3) we have

\[
U_{e_i}(a_k) - \lambda_{ik} e_i \in I_m, \quad (i = 1, \ldots, n, \ k \in \mathbb{N}),
\]
and by continuity of \(d_m\) \(|I_m|\), \(\lim_{k \to \infty} d_m(U_{e_i}(a_k) - \lambda_{ik} e_i) = 0\). Since \(\lim_{k \to \infty} \lambda_{ik} = 0\), we have \(\lim_{k \to \infty} d_m U_{e_i}(a_k) = 0\). Therefore \(d_m U_{e_i}\) is continuous for \(i = 1, \ldots, n\), and \(e_1, \ldots, e_n \in (Q_m)_h\). So \(e_1 + \ldots + e_n \in I_m = (Q_m)_h + i(Q_m)_h\). Since \(\pi(e_1 + \ldots + e_n)\) is the identity of \((\frac{A}{I_m})_h\), \(A = I_m\) and \(d_m\) is continuous on \(A\).

**Lemma 3.7.** Let \(A\) and \(B\) be \(JB^*\) -algebras and let \(\phi: A \to B\) be a \(*\)-homomorphism, that is \(\phi(a^*) = (\phi(a))^*\) \((a \in A)\). Consider \(B\) as a Banach Jordan \(A\)-module via the homomorphism \(\phi\). If \(S\) is a submodule of \(B\), then \(Q(S) = \mathbb{I}(S)\).
Proof. We show that \((Q(S))_h = (I(S))_h\). Consider the identities
\[
(U_x(y^2))^2 = U_x U_y U_y (x^2), \quad (3.4)
\]
\[
(xy)^2 = \frac{1}{2} y U_x (y) + \frac{1}{4} U_x (y^2) + \frac{1}{4} U_y (x^2), \quad (3.5)
\]
which are valid in any Jordan algebra, see [10], p. 37 for the first one. The second holds by the fact that any Jordan algebra generated by two elements is special, see Shirsov-Cohen’s theorem, [7] Theorem 2.4.14. Now, if \(a \in Q(S)_h\) then by setting \(x = \phi(a) \in B_h\) in (3.4) and (3.5), we have
\[
(\phi(a)b)^2 = 0 \quad (a \in Q(S), \ b \in S).
\]
Therefore \(a \in R(S)\) and it follows that \((Q(S))_h \subseteq (R(S))_h\). So \((Q(S))_h = (Q(S))_h \cap (R(S))_h = (I(S))_h\), by ([9], Theorem 1.4).

Corollary 3.8. Let \(A\) and \(B\) be JB*-algebras, and let \(\{d_m\} : A \rightarrow B\) be a higher derivation for which \(d_0\) is a *-homomorphism. Then \(\{d_m\}\) is continuous.

Proof. Since \(d_0\) is a *-homomorphism, it is automatically continuous. Note that \(S_1\) is a submodule of \(B\), thus by Lemma 3.7, \(Q_1\) is a linear subspace of \(A\) and hence by ([9] Theorem 2.2), \(d_1\) is continuous. Fix \(m\), suppose that each \(d_i\) (\(i < m\)) is continuous. Therefore \(S_m\) is a submodule and again by Lemma 3.7, \(Q_m\) is a linear subspace of \(A\), and hence by Theorem 3.6, \(d_m\) is continuous.\(\square\)

In the next few results, by a Jordan higher derivation from a C*-algebra \(A\) we mean a higher derivation from \(A\), with its Jordan product, into a Banach Jordan algebra. Obviously each higher derivation (with respect to the associative product) is also a Jordan derivation. As a consequence of Corollary 3.8 each higher derivation, or each Jordan higher derivation between C*-algebras, is continuous provided that \(d_0\) is a *-homomorphism. In the next results \(d_0\) is not assumed to be a *-homomorphism.

Theorem 3.9. Let \(A\) be a commutative C*-algebra, and let \(B\) be a Banach Jordan algebra. If \(\{d_m\} : A \rightarrow B\) is a Jordan higher derivation such that \(d_0\) is continuous, then \(\{d_m\}\) is continuous.

Proof. By ([9], Theorem 2.4) of, \(d_1\) is continuous. Suppose that \(d_1, \ldots, d_{m-1}\) are continuous. Then \(S_m\) is a submodule. We show that \((Q_m)_h =

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Let \( a \in (Q_m)_h \). We have \( a^2A = aAa = U_aA \subseteq Q_m \), and hence \( a^2A \subseteq Q_m \). Since \( I_m \) is the largest ideal of \( A \) contained in \( Q_m \), \( a^2A \subseteq I_m \). Therefore \( a^4 \in I_m \) and since \( a = a^* \), we have \( a \in I_m \). □

Before proving the next result, we recall that if \( A \) is an associative algebra with associative product \((a, b) \mapsto ab \), and the Jordan product \((a, b) \mapsto ab + ba^2\), then \( U_a(b) = aba \) \((a, b \in A)\).

**Theorem 3.10.** Let \( A \) be a C*-algebra with minimal idempotents, and let \( \{d_m\} \) be a Jordan higher derivation from \( A \) to a Banach Jordan algebra \( B \). If \( d_0, \ldots, d_{m-1} \) are continuous on \( A \), then \( \{d_m\} \) is continuous on \( \text{soc}(A) \).

**Proof.** By ([3], Theorem 30.10), \( \text{soc}(A) \) exists. Let \( M \) denote the set of all minimal idempotents of \( A \). Then

\[
\text{soc}(A) = \sum_{e \in M} eA = \sum_{e \in M} Ae, \tag{3.6}
\]

where by \( \sum \) we mean the algebraic sum. Since \( d_0, \ldots, d_{m-1} \) are continuous, we have

\[
Q_m = \{a \in A : U_ad_m \text{ is continuous}\} = \{a \in A : d_mU_a \text{ is continuous}\}. \tag{3.7}
\]

Suppose that \( a \in \text{soc}(A)_h \), then there exist \( b_1, \ldots, b_n \in A \), and \( e_1, \ldots, e_n \in M \) such that \( a = e_1b_1 + \ldots + e_nb_n \), and hence \( a^* = b_1^*e_1^* + \ldots + b_n^*e_n^* = a \).

So

\[
U_a(b) = aba = \sum_{i=1}^{n} \sum_{j=1}^{n} e_ib_ib_j^*e_j^* \quad (b \in A). \tag{3.8}
\]

We know that the adjoint of a minimal idempotent is also a minimal idempotent, hence by ([3], Theorem 31.6), \( \dim(e_iAe_j^*) \leq 1 \), for \( i, j = 1, \ldots, n \). By (3.8) we have, \( U_a(A) \subseteq \sum_{i=1}^{n} \sum_{j=1}^{n} e_iAe_j^* \), thus \( \dim(U_a(A)) < \infty \) and \( d_m \) is continuous on \( U_a(A) \). This shows that \( d_mU_a \) is continuous on \( A \), and hence by (3.7), \( a \in Q_m \). It follows that \( \text{soc}(A)_h \subseteq Q_m \), and since \( Q_m \) is closed, \( \overline{\text{soc}(A)_h} \subseteq Q_m \). By (3.6) \( \text{soc}(A) \) is an \( * \)-ideal, hence \( \overline{\text{soc}(A)} = \overline{\text{soc}(A)_h} \). Now the same argument as in Theorem 3.4 (iii) implies that \( d_m \) is continuous on \( \text{soc}(A) \). □

**Corollary 3.11.** If \( A \) is a C*-algebra with minimal idempotents such that \( \text{soc}(A) = A \), then each Jordan higher derivation from \( A \) into a Banach Jordan algebra \( B \) with continuous \( d_0 \) is continuous. In particular,
if $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the $C^*$-algebra of all compact operators on a Hilbert space $\mathcal{H}$, then every Jordan higher derivation from $\mathcal{A}$ into a Banach Jordan algebra $\mathcal{B}$ with continuous $d_0$, is continuous.

**Proof.** By the hypothesis, $d_0$ is continuous on $\mathcal{A}$. Suppose by induction that $d_0, \ldots, d_{m-1}$ are continuous on $\mathcal{A}$. Then by Theorem 3.10, $d_m$ is continuous on $\text{soc}(\mathcal{A}) = \mathcal{A}$. The last assertion follows by the fact that $\text{soc}(\mathcal{K}(\mathcal{H}))$ is $\mathcal{F}(\mathcal{H})$, the ideal of finite rank bounded operators on $\mathcal{H}$, which is dense in $\mathcal{K}(\mathcal{H})$. □

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