Seismic Wave Propagation in a Multi-Layered Geological Region, Part II: Transient Case

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ABSTRACT: The Two-dimensional problem of the transient wave propagation in elastic multi-layered half-space is studied by the Direct Boundary Integral Equation Method (DBIEM) combined with the finite difference procedure applied to the time variable. By means of the Wilson-θ method the equations of motion are transformed into a set of elliptic partial differential equations, and then, the DBIE-procedure is applied. The present hybrid formulation employs the fundamental solution depending neither on the frequency nor on the time variable. This is the main advantage of the proposed method. The theoretical seismograms in the time domain are obtained on the free surfaces of two real geological situations for a multi-layered soil region with existence of salt ore deposits.

Keywords: Multi-layered geological region; Transient wave problem; DBIEM together with Wilson-θ method

1. Introduction

Transient waves are generated by the body force \( f(r,t) \) in the Navier-Cauchy equation of motion for elastic solids, or the surface displacement \( u(r,t) \), or traction \( p(r,t) \) given as boundary conditions for the corresponding boundary-value problem. If the time function for these sources is time-harmonic and the motion is observed long after the initiation of the source, the wave motion is also harmonic in time and it is called steady state. Otherwise, the wave motion is transient. In an infinite medium, the transient wave field generated by a concentrated force of arbitrary time function was determined by G. Stokes in 1849. He constructed the general solution of the inhomogeneous Navier-Cauchy equation of motion by what is now called the method of retarded potentials. Next in mathematical complexity is the problem of transient waves in a half-space. H. Lamb in 1904 investigated this problem first. The “half-space” and related problems have since become a focal point for many studies. Many applications of elastodynamics in seismic mechanics, design of earthquake-resistant structures, dynamics of structural foundations as well as basic studies on dynamics of material defects begin with the model of a half-space. The solution is much more complicated when additional plane boundaries are introduced to form a layered half-space. This is a basic model for theoretical seismograms, which has always been a subject of intensive study.

The main aims of this paper are:

- To find the solution of a two-dimensional plane-strain transient wave propagation problem for a multi-layered geological region with complex geometry on the base of a hybrid usage of both finite differences scheme of Wilson-θ method together with the BIEM.
- To show that the changes in the soil region during the years of the exploitation process lead to the change in its dynamic response, i.e. to the change in the obtained theoretical seismograms.

The propagation of transient elastic waves through the layered half-space is of considerable interest to engineers, geologists and seismologists. Lacking any analytical method to treat such complex problems, resort has been made to the numerical techniques-finite element method and boundary integral equation method. Most of the earlier works on transient wave propagation by BEM involved transform domain formulations in conjunction with numerical inversion scheme [8-12], etc. The direct time-domain formulation of the BIEM is used in [13-16], etc. Detail analysis for advantages and disadvantages of BIEM in comparison with other numerical methods is given in [16]. In this paper a BIEM formulation, different from the above-cited two formulations, is proposed. By means of the finite difference method, applied to the time
variable, the equations of motion are transformed into a set of elliptic partial differential equations, and then, the DBIE-procedure is applied at every time step.

The most of the works are devoted to the multi-layered regions with simple geometry of the boundary between layers (usually parallel boundaries). To the authors opinion there is a lack of studies involving real multi-layered regions with complex geometry of the boundaries between layers (non-parallel boundaries).

It is given here BIEM formulation, novel numerical scheme and FORTRAN codes created for solution of a 2D transient seismic waves propagation problem in a multi-layered soil region with very complex geometry of the soil layers and this geological column is a real geological region in East Bulgaria.

2. Formulation of the Problem

2.1. Governing Equations

The mathematical theory of elasticity is formulated in terms of body force $f_j$, surface force (traction) $p_i$, stress tensor $\sigma_{ij}$, strain tensor $\varepsilon_{ij}$ and displacement vector $u_i$ of an elastic body. The isotropic materials are characterized by material constants such as the shear module $G$, Lame’s constants $\lambda$, $\mu$ and mass density $\rho$. The linear theory of elasto-dynamics is embodied in the following set of governing equations for a body of volume $V$ enclosed by a surface $S = S_v + S_p$

**Equation of motion**

$$\sigma_{ij,j} + f_i = \bar{u}_i, \quad x \in V$$

**Physical equations in elastic case**

$$\sigma_{ij} = 2\mu(\nu/(1-2\nu))\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \quad x \in V$$

here: $\nu = \frac{\lambda}{2(\lambda+\mu)}$

**Equation of geometry**

$$\varepsilon_{ij} = \frac{1}{2}(u_{ij,j} + u_{ji,j})$$

The governing equations of the two-dimensional motion of an isotropic and homogeneous elastic medium are obtained by Eqs. (1), (2) and (3) and have the form

$$\left(C_p^2 + C_s^2\right)u_{ij,j} + C_s^2u_{ji,j} - \ddot{u}_j = -\rho^{-1}f_j(t)$$

where $i, j = x, y$, the summation convention applies to the repeated suffix $i$, the notation $(\cdot)_{,j} = \partial(\cdot)/\partial x_j$ is introduced to denote the partial derivatives with respect to co-ordinates $x_j$, the dot signifies material time differentiation and $C_p = [(\lambda + 2\mu)/\rho]^{1/2}$; $C_s = (\mu/\rho)^{1/2}$ are speeds of the longitudinal $P$- and shear $S$-wave, respectively. The problem is two-dimensional and of plane strain and $u_j(x, y, t), \ u_j(x, y, t)$ denote the horizontal and the vertical displacement components respectively.

**The initial and boundary conditions are**

$$u_i(x, y, t) = \bar{u}_i \text{ at } (x, y) \in S_v, \ \sigma_{ij}n_j = \bar{p}_i \text{ at } (x, y) \in S_p$$

$$u_i(x, y, t_0) = u_{i0}, \ \dot{u}_i(x, y, t_0) = \ddot{u}_{i0} \text{ at } (x, y) \in S_v$$

(5)

The governing equations (4) together with the boundary and initial conditions (5) present the boundary-value problem of the transient elasto-dynamics.

2.2. BIE Formulation of the Problem

The considered problem can be solved using the following approaches:

- Laplace or Fourier transformation leading to the next BIE

**c_iu_i(r, \omega) = \oint [U_{ij}(r, \eta, \omega)\tilde{p}_j(r, \eta, \omega) - P_{ij}(r, \eta, \omega)\tilde{u}_j(r, \eta, \omega)]dS$$

$$+ \int U_{ij}(r, \eta, \omega)\tilde{Q}_{ij}(r, \eta, \omega)dV$$

Here $r$ and $\eta$ are the position vectors of the field and running points; the constants $c_{ij}$ depend on the geometry of the boundary at the collocation point $r$, $U_{ij}$ and $P_{ij}$ are the fundamental solutions of the displacement and the traction given in the Appendix; $\tilde{u}_i$, $\tilde{p}_i$ are the corresponding Laplace or Fourier transformations of the displacement and the traction; $\tilde{Q}_{ij}$ is the sum of the Laplace or Fourier transformations of $f_i$ and the members, containing the initial velocity and displacement; the Laplace transformation variable is $S = -j\omega$. This method is used by the authors in [1, 2, 3].

- A hybrid method consists of two stages:

  i) Application of the method of finite differences in respect to the time variable-then the set of parabolic partial differential equations is transformed into a set of elliptic partial differential equations;

  ii) The BIEM is applied at each time step. This hybrid formulation is used by authors in [4] in the simple case of an elastic half-space. This approach will be used in the current paper but in the case of a multi-layered geological region with non-parallel boundaries of the layers. Due to the fact that this method is applied here in a multi-layered region with complex geometry, we will describe it in the next section.

3. A Hybrid Method of Finite Differences Method and BIEM for the Solution of the Transient Elasto-Dynamic Problem

The application of the Wilson-$\theta$ method [5] to $\dot{u}_i$ and $\ddot{u}_i$ yields
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\[
\dot{u}_i(t + \Delta t) = \frac{3}{\theta \Delta t} [u_i(t + \theta \Delta t) - u_i(t)] - 2 \dot{u}_i(t)
\]

\[
\ddot{u}_i(t + \theta \Delta t) = \frac{6}{\theta^2 \Delta t^2} [u_i(t + \theta \Delta t) - u_i(t)] - \frac{6}{\theta \Delta t} \ddot{u}_i(t)
\]

where \( \Delta t \) denotes the time step and \( \theta \) is a coefficient, which secures the stability and the convergence of the finite difference procedure. This procedure becomes unconditionally stable for \( \theta > 1.37 \). After substitution of Eq. (7) in Eq. (4), the next system of elliptic linear partial differential equations is obtained at time \( t + \theta \Delta t \)

\[
(C_p^2 - C_s^2)\dot{u}_j(t + \theta \Delta t) + C_s^2 \dot{u}_j(t + \theta \Delta t) - k^2 u_i(t + \theta \Delta t) = -X_i
\]

where

\[
k^2 = \frac{6}{\theta^2 \Delta t^2}, \quad X_i = p^{-1} f_i(t + \theta \Delta t) + \frac{6}{\theta \Delta t} \ddot{u}_i(t)
\]

As it can be seen from Eqs. (8) the last are the same as the elliptic partial differential equations, obtained after the application of the Laplace transform with \( k^2 = \frac{6}{\theta^2 \Delta t^2} \). That is why the conventional direct BIEM can be applied to Eqs. (8) at a given time step \( \Delta t \). The fundamental solutions, given in the Appendix, can be used, but in this case \( s = -\omega_0 = k \). The discretization used in the analysis involves \( N \) constant boundary elements and \( M \) constant triangular elements. The points, where the unknown values are considered, are called "nodes" and they are in the middle of the element for the so-called "constant" element. The values of the displacement and the traction are assumed to be constant over each element and equal to the value at the mid-element node. For this type of elements the boundary is always "smooth" as the node is at the centre of the element. The discretization nodes are in the middle points of the all \( N \) boundary elements over the free surface and they are in the centres of gravity of the all \( M \) constant triangular elements used for discretization of the half-space. As a result the next system of discretized boundary integral equations is obtained

\[
\begin{bmatrix}
0 & f(t)
\end{bmatrix}
\begin{bmatrix}
u_x(x_m, y_m)
\end{bmatrix}
= \begin{bmatrix}
A_x(x_m, y_m)
B_x(x_m, y_m)
C_x(x_m, y_m)
\end{bmatrix}
\begin{bmatrix}
u_x(x_m, y_m)

\end{bmatrix}
\begin{bmatrix}
U_x(x_m, y_m)
U_y(x_m, y_m)
\end{bmatrix}
\begin{bmatrix}
p_x(x_m, y_m)
p_y(x_m, y_m)
\end{bmatrix}
\]

where \( f(t) \) denotes the time signature of the source, \( c_{xx} = c_{yy} = 1 \) for the interior points, \( c_{xx} = c_{yy} = 0.5 \) for the points, located at the boundary

\[
A_x(x_m, y_m) = \sum_{k=1}^{N} \int_{\Gamma_k} U_x(x_m - x_0, y_m - y_0) d\Gamma_k
\]

\[
B_x(x_m, y_m) = \sum_{k=1}^{N} \int_{\Gamma_k} U_y(x_m - x_0, y_m - y_0) d\Gamma_k
\]

\[
C_x(x_m, y_m) = \sum_{k=1}^{N} \int_{\Gamma_k} p_x(x_m - x_0, y_m - y_0) d\Gamma_k
\]

\[
D_x(x_m, y_m) = \sum_{k=1}^{N} \int_{\Gamma_k} p_y(x_m - x_0, y_m - y_0) d\Gamma_k
\]

\[
\alpha_k = k u_k(x_m, y_m, t) + \frac{6}{\theta \Delta t} \ddot{u}_k(x_m, y_m, t) + 2 \dddot{u}_k(x_m, y_m, t)
\]

Note, that Eqs. (9a - 9c) are evaluated at time \( t + \theta \Delta t \), while Eq. (9d) is evaluated at time \( t \). In the above matrices \( u_x(x_m, y_m) \) and \( u_y(x_m, y_m) \) are components of the unknown displacement, \( p_x(x_m, y_m) \) and \( p_y(x_m, y_m) \) are components of the unknown traction, \( x_S, y_S \) are the co-ordinates of the seismic source, and the notations \( \partial / \partial x_S \) and \( \partial / \partial y_S \) are introduced to denote the partial derivatives with respect to \( x_S \) and \( y_S \) respectively.
4. Numerical Realization Algorithm

The numerical procedure for the solution of the transient elasto-dynamic problem described above is:

- Solve the system of Eq. (9) for \( t = 0 + \theta \Delta t \) and obtain \( u_x, u_y, p_x, p_y \) at all collocation points on the boundary (note that for \( t = 0, \alpha^k_x = 0 \)).
- For \( t = 0 + \theta \Delta t \) evaluate \( u_x, u_y, p_x, p_y \) and \( \alpha^k_x, \alpha^k_y \) at all interior collocation points.
- For \( t = 0 + \Delta t \) by the Wilson-\( \theta \)-method calculate \( u_x, u_y, \ddot{u}_x, \ddot{u}_y \).
- For \( t = \Delta t + \theta \Delta t \) solve the system of Eq. (9) and obtain \( u_x, u_y, p_x, p_y \) at all collocation points on the boundary.
- For \( t = \Delta t + \theta \Delta t \) evaluate \( u_x, u_y, p_x, p_y \) and \( \alpha^k_x, \alpha^k_y \) at all interior collocation points.
- For \( t = 2\Delta t \) by the Wilson-\( \theta \)-method calculate \( u_x, u_y, \ddot{u}_x, \ddot{u}_y \).
- For \( t = 2\Delta t + \theta \Delta t \) solve the system of Eq. (9) and obtain \( u_x, u_y, p_x, p_y \) at all collocation points on the boundary.
- For \( t = 2\Delta t + \theta \Delta t \) evaluate \( u_x, u_y, p_x, p_y \) and \( \alpha^k_x, \alpha^k_y \) at all interior collocation points.

Based on the above scheme, the evaluation of responses for \( t = 3\Delta t + \theta \Delta t \) and subsequent observation times is straightforward.

5. Numerical Example

Two real geological situations for a multi-layered soil media with existence of salt ore deposits, in Figures (1) and (2) are considered. These situations concern one and the same geological region but in different periods of its exploitation in 1951, see Figure (1) and 1994, see Figure (2). Due to the symmetry the half of the geometry is given correspondingly in Figures (3) and (4). There is a change of the situation during the years when the exploitation of the salt ore deposits has been done. The main goal is to show that the changes in the soil region during all these years lead to the change in its dynamic response, i.e. to the change in the obtained theoretical seismograms. It is assumed that the region is subjected to the buried explosive seismic load, which time function is the parabolic ramp function given in Figure (5).

\[
 f(t) = \begin{cases} 
 0 & \text{for } t < 0 \\
 0.5t^2 & \text{for } 0 \leq t \leq \Delta \\
 0.5t^2 - (t-\Delta)^2 & \text{for } \Delta \leq t \leq 2\Delta \\
 \Delta^2 & \text{for } t \geq 2\Delta 
\end{cases} 
\]

\[ (10) \]

\[ (10) \]

Figure 1. The (2D) geometry of the multi-layered geological region in 1951.

Figure 2. The (2D) geometry of the multi-layered geological region in 1994.

Figure 3. A half of the geometry of the geological region in 1951.
Two-dimensional in-plane transient wave propagation problem will be solved for the above geological columns. The plane strain state is considered. The soil material is homogeneous, isotropic and elastic. The displacement vector is with components $u_x(x, y, t)$ and $u_y(x, y, t)$.

The initial conditions are

$$u_i(x, y, t_0) = u_i^0(x, y)$$

$$\dot{u}_i(x, y, t_0) = \dot{u}_i^0(x, y)$$

(11)

The boundary conditions are, see Figures (3) and (4).

It is convenient to represent the motion in the half-space as a superposition of the “free-field” motion and the waves scattered from the multi-layered geological column. The “free-field” motion consists of the plane incident wave and the reflected waves of the free boundary. The solution of the problem for transient elastic waves in a half-space without any layer due to the buried explosive load of type (10) is given and compared with other solutions in [4].

The governing Eqs. (4), the initial conditions (11) and the boundary conditions (12) represent the considered boundary-value problem. The problem is solved following the numerical procedure described in detail in section 4.

Following the above-described numerical procedure, overcoming of weak and strong singularities in the obtained integrals and after satisfaction of the given boundary conditions, an algebraic complex system according to the unknowns is obtained.

The regular integrals are computed numerically employing the Gaussian quadrature scheme. The kernels of the type $\int_{\mathcal{K}} k_{ij} \, d\Gamma$ have singularities like $O(\frac{1}{c \pm \xi})$ for $c \in [-1, +1]$ that leads to the CPV integrals. The kernels of the type $\int_{\mathcal{K}} \dot{k}_{ij} \, d\Gamma$ have singularities like $O(\ln(c \pm \xi))$ for $c \in [-1, +1]$, which leads to non-singular integrals. The singular integrals are solved analytically based on the asymptotic expansion for a small argument of the Bessel function. In all corner points, where the Helder continuity conditions are not satisfied, it is applied the Lachat and Watson concept [7] that $p_{ij}^{-1} - \sigma_i n_j = p_{ij}^s - \sigma_i n_j$, for $p_{ij}^s$, $\sigma_i$, $n_j$. The influence of the geological column on the motion of the half-space has to vanish at sufficiently large distances, or Sommerfield boundary conditions have to be satisfied at infinity.

On the boundary between two soil layers $\Omega_i$ and $\Omega_{i+1}$ inside the geological column the next continuity conditions have to be satisfied

$$u_i(x, y)_{\Omega_i} = u_i(x, y)_{\Omega_{i+1}}$$

(12c)

and the motion must be such that all the dynamic forces acting onto the boundary are in dynamic equilibrium

$$p_i(x, y)_{\Omega_i} = p_i(x, y)_{\Omega_{i+1}}$$

(12d)

On the boundary between the geological column and the half-space the next boundary conditions have to be satisfied

$$u_i(x, y)_{e} = u_i^{free-field}(x, y)_{e}$$

(12e)

for $(x, y) \in AB, BC, CE, EF, e = \Omega_1, \Omega_4, \Omega_5$ for Figure (3) and $e = \Omega_2, \Omega_4, \Omega_5$ for Figure 4. The displacement field $u_i^{free-field}(x, y)$ is obtained after the solution of the problem for transient elastic waves in a half-space without any layer due to the buried explosive load of type (10).

The governing Eqs. (4), the initial conditions (11) and the boundary conditions (12) represent the considered boundary-value problem. The problem is solved following the numerical procedure described in detail in section 4.

Following the above-described numerical procedure, overcoming of weak and strong singularities in the obtained integrals and after satisfaction of the given boundary conditions, an algebraic complex system according to the unknowns is obtained.

The regular integrals are computed numerically employing the Gaussian quadrature scheme. The kernels of the type $\int_{\mathcal{K}} k_{ij} \, d\Gamma$ have singularities like $O(\frac{1}{c \pm \xi})$ for $c \in [-1, +1]$ that leads to the CPV integrals. The kernels of the type $\int_{\mathcal{K}} \dot{k}_{ij} \, d\Gamma$ have singularities like $O(\ln(c \pm \xi))$ for $c \in [-1, +1]$, which leads to non-singular integrals. The singular integrals are solved analytically based on the asymptotic expansion for a small argument of the Bessel function. In all corner points, where the Helder continuity conditions are not satisfied, it is applied the Lachat and Watson concept [7] that $p_{ij}^{-1} - \sigma_i n_j = p_{ij}^s - \sigma_i n_j$, for $p_{ij}^s$, $\sigma_i$, $n_j$.
two different boundary elements $\Gamma^{s-1}$ and $\Gamma^s$, which formed the corner.

6. Numerical Results

In this item the numerical results obtained after the solution of the boundary-value problem, formulated in section 5, using the numerical scheme which is described in section 4 based on the Direct Boundary Integral Equation Method combined with the finite difference procedure applied to the time variable are given. The advantages of this novel BIEM formulation are discussed in the above items.

The geometrical characteristics of the geological regions, presented in Figures (1) and (3) and Figures (2) and (4) are given in Tables (1) and (2). The mechanical properties of these geological columns are shown in Tables (1) and (2), see part I of the paper.

The depth of the applied dynamical load is $4500 \text{m}$ and its location is $(0.0, 4500 \text{m})$. The epicentre distance of the receiver is $x = 1400 \text{m}$. The time step at the realization of the numerical algorithm is $0.001 \text{s}$ and the value of the parameter $\theta$ is $1.4$.

The theoretical seismograms in the time domain for the horizontal $u_x$ and vertical $u_y$ components of displacement at the receiver, located at the free surface of the layered half-space at distance $x = 1400 \text{m}$, are obtained. They are perturbed by a buried explosive seismic source (that is a line source of dilatation acting at $x_S = (0.0, 4500)$). The time function of the source $f(t)$, see Eq. (10) is assumed to be parabolic ramp function which half-rise time $\Delta$ is taken as $\Delta = 0.1 \text{s}$.

Figures (6a)-(6b) show the horizontal and vertical component of displacement in the time domain at a point $(1400, 0.0)$, when the boundary-value problem is solved for the geological region given in Figure (1).

The theoretical seismograms in the time domain for the horizontal and vertical displacements obtained at a point $(1400, 0.0)$ when the boundary-value problem is solved for the geological region given in Figure (2) are shown in Figures (7a)-(7b).

One can see that the transient dynamic responses of one and the same real geological region - a multi-layered

| Table 1. Geometrical parameters of the geological column N1. |
|-----------------|----------------|
| **Boundary**    | **Length [m]** |
| AB              | 3400           |
| BC              | 1000           |
| CE              | 2000           |
| EF              | 1000           |
| FI              | 3000           |
| IJ              | 400            |
| JK              | 1000           |
| KL              | 500            |
| LA              | 2500           |

| Table 2. Geometrical parameters of the geological column N2. |
|-----------------|----------------|
| **Boundary**    | **Length [m]** |
| AB              | 3400           |
| BC              | 1000           |
| CE              | 2000           |
| EF              | 1000           |
| FI              | 3000           |
| IJ              | 400            |
| JK              | 500            |
| KL              | 500            |
| LQ              | 350            |
| QS              | 150            |
| SW              | 200            |
| WA              | 2300           |

Figure 6a. Theoretical seismogram for the horizontal displacement in column N1.

Figure 6b. Theoretical seismogram for the vertical displacement in column N1.
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soil media with existence of salt ore deposits, obtained at different exploitation periods (in 1951 and in 1994) are different. The change of the geological situation during the years due to the exploitation process in the salt ore deposits leads to the different transient responses obtained at the application of a buried dynamic transient load.

7. Conclusion

An application of the novel two-dimensional formulation of the direct boundary integral equation method proposed in [4] has been applied here for multi-layered geological columns with complex geometry. This application concerns two real geological situations for multi-layered soil media with existence of salt ore deposits. Figures (1) and (2) show one and the same geological region but in different periods of time. One can see that the change of the geological situation during the years when the exploitation of the salt ore deposits has been done leads to the change of the transient wave picture due to the buried dynamic load. The time records of the surface responses are computed by the proposed in [4] novel formulation of DBIEM for the real geological situations and different wave fields are obtained. These results show that the changes in the soil region lead to the change in its dynamic response, i.e. to the change in the obtained theoretical seismograms. All this assures us that the exploitation process leads to the changes in the geological situation of the region, respectively to the changes of the soil response during eventual earthquake.

Finally we would like to underline once again that the present hybrid formulation, combined the finite difference procedure with the direct boundary integral equation method employs the fundamental solution depending neither on the frequency nor on the time variable. This is a serious advantage of the proposed method. This work shows that the method proposed here works well even in the cases of multi-layered geological regions with complex geometry.*

Aknowledgments

The support of the National Science Foundation of U.S.A. grant DMS97-09320 is gratefully acknowledged.

References


* Part of these results is reported in the SEE-3 Conference, May 17-19, 1999, Tehran, I.R.Iran [6].
the Third Int. Conference on Seismology and Earthquake Engineering (SEE-3), Tehran, Iran, 1, 523-530.


Appendix

The function \( U^\gamma_{ij}(x,y,x_0,y_0,\omega) \) is the fundamental solution of the system of Eq. (15) in Part I of this paper and the traction function \( P^\gamma_{ij}(x,y,x_0,y_0,\omega) \) is the corresponding traction function

\[
U^\gamma_{ij}(x^\gamma-x^\delta,y^\gamma-y^\delta,\omega) = \frac{1}{2\pi i} \left( -X_{ij} - X_{ij} \right) \int \frac{V^\gamma_{ij}}{V^\delta_{ij}} \left( \delta_{ij} + \frac{\partial}{\partial r} \right) \left( \delta_{ij} + \frac{\partial}{\partial n} \right) \]

\[
P^\gamma_{ij}(x^\gamma-x^\delta,y^\gamma-y^\delta,\omega) = \frac{1}{\pi i} \left( -Y_{ij} - Y_{ij} \right) \int \frac{V^\gamma_{ij}}{V^\delta_{ij}} \left( \delta_{ij} + \frac{\partial}{\partial r} \right) \left( \delta_{ij} + \frac{\partial}{\partial n} \right) \]

where \( (x^\gamma, y^\gamma) \) and \( (x^\delta, y^\delta) \) are the field point and the running point respectively

\[
r = \sqrt{(x^\gamma - x^\delta)^2 + (y^\gamma - y^\delta)^2}
\]

\[
r = \frac{r_x}{r} = \frac{x^\gamma - x^\delta}{r}
\]

\[
r = \frac{r_y}{r} = \frac{y^\gamma - y^\delta}{r}
\]

\[
\partial_r x = n_q + r \cdot n_q \cdot s = -i\omega
\]

\[
\chi = K_2 \left( \frac{sr}{V_s} - \frac{V_s^2}{V_P} K_2 \left( \frac{sr}{V_P} \right) \right)
\]

\[
\psi = \frac{1}{2} \left[ K_2 \left( \frac{sr}{V_s} - \frac{V_s^2}{V_P} K_2 \left( \frac{sr}{V_P} \right) \right) + \frac{1}{2} \left[ K_2 \left( \frac{sr}{V_s} + \frac{V_s^2}{V_P} \right) + \frac{V_s^2}{V_P} K_0 \left( \frac{sr}{V_P} \right) \right] \right]
\]

The function \( K_m(z) = -\frac{1}{2} i \pi^{m-1} H_m^{(2)}(\xi) \) for \( z = -i\omega \)

or \( z = -i\omega V_1 \) is represented with the modified Bessel functions of the second type.

The asymptotic representations of the functions \( U^\gamma_{ij} \) and \( P^\gamma_{ij} \) for \( r \to 0 \) are

\[
(U^\gamma_{ij})^w = -\frac{1}{4\pi i} \left[ \left( 1 + \frac{V_s^2}{V_P} \right) \left( n_r + \ln \frac{V_s}{V_P} \right) \frac{V_s^2}{V_P} \delta_{ij} - A \right]
\]

\[
(P^\gamma_{ij})^w = -\frac{1}{2\pi i} \frac{V_s^2}{V_P} \left[ \delta_{ij} + 2 \left( 1 + \frac{V_s}{V_P} \right) \frac{V_s}{V_P} \frac{\partial}{\partial n} \right] A
\]

where

\[
A = -\frac{1}{4\pi i} \left[ \left( 1 - \frac{V_s^2}{V_P} \right) \frac{\partial}{\partial n} + i \frac{3\pi}{2} \left( 1 + \frac{V_s^2}{V_P} \right) \right]
\]

\[
BB = -\frac{1}{2\pi i} \frac{V_s^2}{V_P} \left( n_r - n_j \right) A
\]