RESEARCH NOTE

M/M/m/K QUEUE WITH ADDITIONAL SERVERS AND DISCOURAGEMENT

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Abstract Waiting in long queue is undesirable due to which in some practical situations customers become impatient and leave the system. In such cases providing additional servers can reduce long queue. This investigation deals with multi-server queueing system with additional servers and discouragement. The customers arrive in poisson fashion and are served exponentially by a pool of permanent and additional servers on the basis of FCFS queue discipline. The explicit formula to determine the number of customers in the queue has been obtained. Numerical illustration has been provided to validate the analytical results established. Graphs are drawn for expected number of customers in the system and probability of all additional servers being busy in order to visualize the effect of various parameters.

Key Words Multi-Server Queue, Finite Capacity, Additional Servers, Queue Size, Discouragement, Cost Analysis

1. INTRODUCTION

For an organization, the service costs recommend a minimal level of service, whereas undesirable long waiting times recommend a high level of service. From the cost viewpoint these two considerations create conflicting pressure on the decision maker. Therefore it is necessary to evolve some way of compromise so that queuing model may give the best balance between the average delay and the cost of providing this service. Providing additional servers for a long queue can do this. Murari [1] studied a limited waiting space queueing system with additional special channel for a long queue. Makaddis and Zaki [2] developed the M/M/1/∞ queuing system with additional servers for a long queue. Varshney et al. [3] extended this work to M/M/m/K queuing system with additional servers. Jain [4] suggested the provision of additional repairmen apart from permanent repair facility for M/M/R machine repairmen problem with spares. Jain and Ghimire [5] studied M/M/m/K queue with no-passing and additional servers.

It is obvious that long waiting times are undesirable due to which in some real world problems, customers get impatient and leave the
system, which causes the loss to the system and also inconveniences to the customers. Jain et al. [6] introduced concept of nopassing for multiserver queueing model with two types of customers and discouragement.

Some efforts have been made to study queueing models with discouragement via diffusion approximation. Varshney et al. [7] analyzed the diffusion approximation for G/G/m queueing system with discouragement. Garg et al. [8] developed the G/G'/m queueing system with discouragement via diffusion approximation. Jain [9] and Varshney et al. [10] investigated queueing model with discouragement via diffusion approximation and provided some useful results for the measures of effectiveness. Jain and Singh [11] studied a finite capacity priority queueing model with discouragement. Jain [12] investigated Markovian queueing model with discouragement and having the facility of additional servers to share the load in case of a long queue. The customers were assumed to be originated from infinite source and there is provision of infinite waiting space. But in realistic situations, the assumption of infinite capacity is unrealistic so that the study of finite capacity model is more desirable. Recently, Jain et al. [13] discussed M/M/C/K/N machine repair problem with balking, reneging, spares and additional repairman.

In this study, we investigate the finite capacity M/M/m/K queue with additional servers and discouragement. The paper is organized as follows. Mathematical model and analysis are given in section 2. The expected number of customers in long run and the probability that a particular server being busy have been derived in section 3. Section 4 provides particular cases wherein comparative study for model with and without additional servers has also been documented. For illustration purpose, some numerical results have been reported in section 5. In last section 6, we conclude our investigation by discussing the various applications of the model.

2. MATHEMATICAL MODEL AND ANALYSIS

The customers arrive according to Poisson process with mean rate \( \lambda \) and are served exponentially with mean rate \( \mu \) by a pool of \( s \) including \( r = s-m \) additional servers on the basis of FCFS queue discipline. The number of servers required for service depends upon the number of customers present in the system in the following ways:

- When the number of customers is less than or equal to \( N \), \( m (\geq 1) \) permanent servers will be available in the system.
- When the number of customers is greater than \( N \) and less than or equal to \( 2N \), an additional server will be provided in the system. In general, when the number of customers is greater than \( jN \) and less than or equal to \( (j+1)N, j = 1, 2, 3, ..., (s-m-1) \), system will provide \( j \) additional servers. The \( j \)th (\( j = 1, 2, 3, ..., s-m \)) additional server will be turned off as soon as the number of customers in the system drops to \( jN \).
- There will be all \( r \) additional servers available when there are more than \( rN \) customers in the system.

The customers are assumed to be discouraged by a long queue so that the arrival rate depends on the number of customers per server present in the system. The state-dependent arrival and service rates for the model are given by

\[
\lambda_n = \begin{cases} 
\lambda & ; n < m \\
\left( \frac{m}{n+1} \right)^\beta \lambda & ; m \leq n < N \\
\left( \frac{m+j}{n+1} \right)^\beta \lambda & ; jN \leq n < (j+1)N \\
\left( \frac{s}{n+1} \right)^\beta \lambda & ; (s-m)N \leq n < K 
\end{cases}
\]  

(1)

where \( \beta \) is a positive constant called "pressure coefficient" which indicates the degree to which the arrival rate is effected by the number of customers per server in the system. Thus, the customers balk with probability \( 1 - \left( \frac{m+j}{n+1} \right)^\beta \) where there are \( n \) customers in the system and \( j (=1, 2, ..., s-m) \) additional servers are available.
Let $P_n$ be the steady state probability of being $n$ customers in the system. Using Equations 1 and 2 and birth death process [14], we get

$$P_n = \begin{cases} 
\frac{\rho^n}{n!} & ; 0 \leq n \leq m \\
\frac{\rho^n}{n!} \frac{(m+\beta)(m-n)^{m-n}}{m!} & ; m \leq n < N \\
\frac{\rho^n}{n!} \frac{(m+\beta)(m-n)^{m-n}}{m!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} & ; jN \leq n \leq (j+1)N \\
\frac{\rho^n}{n!} \frac{(m+\beta)(m-n)^{m-n}}{m!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} & ; (s-m)N \leq n \leq K 
\end{cases}$$

(2)

where $\rho$ is traffic intensity and is given by relation

$$\rho = \frac{\lambda}{\mu(s-m)}$$

By using normalizing condition $\sum_{n=0}^{K} P_n = 1$ we get

$$P_n = \left[ 1 + \sum_{n=1}^{M} \frac{\rho^n}{n!} \frac{(m+\beta)(m-n)^{m-n}}{m!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} \right] \frac{\rho^n}{n!} \frac{(m+\beta)(m-n)^{m-n}}{m!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}}$$

(3)

### 3. PERFORMANCE MEASURES

Expected number of customers in the long run with $(s - m)$ additional servers is given by

$$E(Q) = P_0 \left[ \sum_{n=0}^{m} \frac{\rho^n}{n!} + \frac{\rho^{m+1}}{m!} \sum_{n=m+1}^{K} \frac{\rho^n}{n!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} \right]$$

$$+ \frac{\rho^m}{m!} \sum_{j=1}^{m-1} \sum_{n=jN+1}^{(j+1)N} \frac{\rho^n}{n!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}}$$

(4)

The probability that the number of customers greater than or equal to $(jN+1)$ and less than $(j+1)N$ in long run is

$$\text{Prob. } [(jN+1) \leq Q < (j+1)N] = P_0 \left[ \frac{\rho^m}{m!} \sum_{j=1}^{m-1} \sum_{n=jN+1}^{(j+1)N} \frac{\rho^n}{n!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} \right]$$

(5)

The probability that $(s-m)N$ in the system is greater than or equal to $(s-m)N$ in the system is

$$\text{Prob. } [(s-m)N \leq Q \leq K] = P_0 \left[ \frac{\rho^m}{m!} \sum_{j=1}^{m-1} \sum_{n=(s-m)N+1}^{(s-m+1)N} \frac{\rho^n}{n!} \frac{1}{\prod_{r=0}^{s-m}(m+r)^{(1-\beta)(n-r)N}} \right]$$

(6)

where $j = 0, 1, 2, ..., (s-m-1)$.

The probability that the number of customers greater than $(s-m)N$ in the system is

$$\text{Prob. } [(s-m)N \leq Q < K]$$

(7)

### 4. SPECIAL CASES

**Case – I** When $\beta = 0$, $P_n$ given by the Equation 3 reduces to results for the M/M/m/K model with additional servers (See Varshney et al. [3])
Case – II For j = 0 and m = s, we get results for a model with discouragement which tally with standard results (cf. [14]).

5. NUMERICAL ILLUSTRATION

To quantitatively characterize the performance of the system, we tabulate the probability of jth additional servers being busy and expected number of customers E(Q) in the system. Table 1 demonstrates the probability of jth (j=1,2,3) additional servers being busy. In table 2, we display E(Q) and the probability of all additional servers being busy by fixing the other parameters as follows: m = 2, K = 50, N = 2, \( \beta = 0.5 \) and \( \mu = 1 \). It is observed that E(Q) decreases as pressure coefficient (\( \beta \)) and number of servers (s) increases which tally with
In figure 1, the effect of arrival rate on the expected number of customers for fixed $\beta = 0.5$, $\mu = 1$, $m = 2$, $N = 2$ and $K = 50$ by varying additional servers $r = 1, 2, 3$ is depicted. We observe that expected number of customers in the system decreases by providing more additional servers. Figure 2 demonstrates the probability of all additional servers being busy for $r = 1, 2, 3$. It is easily seen that probability decreases with the increase in $r$. Figures 3 and 4 display $E(Q)$ and Prob. $\{rN \leq Q < K\}$ vs. $\beta$ for $r = 1, 2, 3$. Figures 3 reveals that as $\beta$ and $r$ increase, $E(Q)$ decreases. From Figure 4, we observe that probability of all additional servers being busy decreases as $\beta$ and $r$ increase.

### 6. DISCUSSION

This model enables us to reduce the waiting times of the customers in the system with the reasonable service cost by providing additional servers in case of a long queue. Admittedly, explicit formula obtained is extremely useful to estimate the number of customers in the queue at any instant directly. The effect of providing additional servers on waiting time plays a significant role to reduce the balking behavior of the customers. In all, this model is of immense importance and has many potential applications in computer, production, telecommunication systems to reduce the waiting time in case of heavy traffic. Some other areas of applications are air/railway reservation, collection counters at various places, particularly on the last day, and priority health centers in rural areas, etc.

### 7. ACKNOWLEDGEMENT

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### 8. REFERENCES


### TABLE 1. Probability of jth additional servers being busy for $\lambda = 3$, $\mu = 1$, $m = 2$, $N = 2$ and $K = 50$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\beta$</th>
<th>$\text{Prob. } [(jN+1) \leq Q &lt; (j+1)N]$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$j=1$</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.514</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.512</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.509</td>
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<tr>
<td>3</td>
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</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.489</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.493</td>
</tr>
<tr>
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<tr>
<td></td>
<td>0.9</td>
<td>0.443</td>
</tr>
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### TABLE 2. The expected number of customers in the system and the Probability of all additional servers being busy for $\lambda = 3$, $\mu = 1$, $m = 2$, $N = 2$ and $K = 50$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$E(Q)$</th>
<th>$\text{Prob. } [rN \leq Q &lt; K]$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$s = 3$</td>
<td>$s = 5$</td>
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<tr>
<td>0.1</td>
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</tr>
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<td>0.4</td>
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<td>0.5</td>
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