Modelling the Nonlinear Deformation in Steel-belted Radial Tyres Under Inflation Loading

Mir Hamid Reza Ghoreishy*
Department of Rubber Engineering and Processing, Iran Polymer and Petrochemical Institute
P.O.Box: 14965/115, Tehran, I.R. Iran
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A new nonlinear axisymmetric finite element model has been developed for the stress-strain analysis of pneumatic tires. The model combines the built-in nonlinear capabilities of the commercial code COSMOS/M (NSTAR module) with tensorial calculations required to compute the stiffness properties of a cord-rubber composite in an arbitrary direction. This model is used for the analysis of a 175/70R14 steel-belted radial tyre under inflation loading. The results showed that ignoring the effect of bias cord angle (e.g., belt angle) on the elastic properties of the cord-rubber composite give rise to prediction of inaccurate tyre deformed shape. The accuracy and validity of the model has also been verified by the comparison of the value of the computed tyre crown displacement with available experimental results.

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INTRODUCTION

Despite the development of advanced finite element techniques for the simulation of the pneumatic tyres under different loading conditions in recent years, axisymmetric analysis is still the most popular and the primary stage in the design cycle of tyres. This is because of the following reasons:

- It requires only a few seconds of CPU time in a PC type computer to predict the inflated shape of the tyre and thus the problem of the tyre sizing can be easily and quickly tackled.
- The quality of the preliminary design can be evaluated without performing highly sophisticated computations, which generally needs advanced computer software and hardware.

- Although tyres undergo wide ranges of static and dynamic loading conditions, the state of stress and strain inside a tyre are built up based on the initial stress and strain fields, which can be predicted by an axisymmetric analysis.

Pneumatic tyres have highly complex structures in which isotropic materials and long fibre composites with orthotropic properties are used together to construct a cohesive and unique shape. The fibres are used in different parts of the tyres such as ply, belt, cap ply and bead. In each part, fibers are oriented in a particular direction to achieve the required physical and mechanical properties. The axis of symmetry is then different from the axis of orthotropy and thus most of the 2-dimensional axisymmetric elements, which are available in commercial nonlinear codes, are unable to take the out-of-plane orientation of the long fibers into account. In the present work we have developed a numerical technique and a computer code to consider the effect of the out-of-plane orientation of long fiber used in the steel belted radial tyres. This code is used to modify the source code of commercial software NSTAR ver 1.75 [1]. The modified code is then employed for the analysis of a 175/70R14 steel-belted radial tyre under inflation pressure load. The accuracy of the developed model and computer code has been verified by the comparison of the computed displacements with experimentally measured data. It is shown that erroneous results are obtained if the out-of-plane cord angle is neglected during the axisymmetric analysis of a pneumatic tyre.

NONLINEAR FINITE ELEMENT FORMULATION

Consider the large deformation of a body under load in a stationary Cartesian coordinate system from an initial configuration to the deformed state. In a nonlinear finite element analysis the equilibrium of the body must be established in the current (deformed) configuration. Assuming that the loads are applied as a function of time, the aim is to predict the equilibrium of the body at discrete time points, where $\Delta t$ is an increment in time. This means that an incremental formulation is employed. In the solution strategy it is assumed that solutions for all time steps from time 0 to time $t$ has been obtained and the solution corresponding to time $t+\Delta t$ is required. Therefore, in order to properly track the deformation of the all particles of the body, a Lagrangian formulation is generally adopted.

Using the principle of virtual displacements, the equation of the equilibrium of a body at time $t+\Delta t$ in a Lagrangian incremental framework is expressed as [2]:

$$\int \tau_{ij} \delta_{ij} e_{ij} \, dV = -\Delta t \mathbf{R}$$  \hspace{1cm} (1)

where $\tau_{ij}$ are the Cartesian components of the Cauchy stress tensor, $e_{ij}$ are infinitesimal strain tensor referred to the configuration of the body at time $t+\Delta t$ and $\delta$ is the variation defined as:

$$\delta_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$  \hspace{1cm} (2)

In this equation, $u_i$ and $u_j$ are the components of the displacement vector and $x_i$ and $x_j$ are the components of the Cartesian coordinates. The right hand side of eqn (1) corresponds to the external virtual work expressed by:

$$\Delta t \mathbf{R} = \int \mathbf{f}^B \delta u_i \, dV + \int \mathbf{f}^S \delta u_i \, dS$$  \hspace{1cm} (3)

where, $\mathbf{f}^B$ and $\mathbf{f}^S$ are the components of the externally applied body and surface force vectors referred to time $t+\Delta t$, respectively.

Since the configuration of the body changes continuously in a large deformation analysis, appropriate stress and strain measures as well as constitutive relations should be used. This is because that the internal virtual work in eqn (1) should be expressed in terms of an integral over a known volume and to be able incrementally decompose stresses and strain in an effective manner. The stress used for the nonlinear structural finite element analysis is 2nd Piola-Kirchhoff stress. At time $t$ this stress is referred to configuration of the body at time 0 and defined as:

$$\boldsymbol{S}_{ij} = \frac{\partial \mathbf{h}_m}{\partial x_m} \mathbf{e}_{ij}$$  \hspace{1cm} (4)
The strain tensor used with the 2nd Piola-Kirchhoff stress tensor is the Green-Lagrange strain tensor defined as:

$$\varepsilon_{ij}^0 = \frac{1}{2}(\dot{U}_{ij}^0 + \dot{U}_{ji}^0)$$

(5)

where, $S$ is the stress tensor, $\varepsilon$ is the strain tensor, $\rho$ is the density of the material and $x_{i,m}$ is the element (i,m) of the gradient deformation tensor $\nabla X = (\nabla^T X)^T$. In the above equations a comma denotes differentiation with respect to the coordinate following (e.g., $0x_{i,m} = \partial \dot{x}_i / \partial x_{mn}$), the left subscript indicates the configuration in which this coordinate is measured and the left superscript indicates in which configuration the quantity occurs. Both the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors are symmetric and objective tensors, which means that rigid body motions of the material do not alter their components. Using these new stress and strain measures, the basic eqn (1) can now be expressed by:

$$\int \rho_0 V_0 0 \varepsilon_{ij}^0 dV + \int \rho_0 V_0 0 \delta \Delta \varepsilon_{ij}^0 dV = t + \Delta \mathcal{R}$$

(6)

The above formulation is generally called total Lagrangian (T.L.) formulation in which all static and kinematic variables are referred to the initial configuration at time $t^0$. The present work is based on the use of the T.L. formulation for the considering the large deformation of tyre under loads. The approximate solution to above equation is obtained by linearizing of equation (6), which is given by:

$$\int \rho_0 V_0 C_{ijrs}^0 0 \varepsilon_{tr}^0 \delta \varepsilon_{ij}^0 dV + \int \rho_0 V_0 S_{ij} \delta \eta_{ij}^0 dV = t + \Delta \mathcal{R} - \int \rho_0 V_0 S_{ij} \delta \varepsilon_{ij}^0 dV$$

(7)

where, $C_{ijrs}^0$ is the linear material property tensor, $\varepsilon_{ij}^0$ is the linear incremental strain tensor and $\eta_{ij}^0$ is the nonlinear incremental strain tensor at time $t$ referred to the configuration at time $t^0$, respectively. The right hand side of eqn (7) represents an out-of-balance virtual work, which should be continuously reduced within a certain convergence tolerance by performing iteration during the solution of this equation. The iterative format of eqn (7) for $k=1,2,\ldots$ is

$$\int \rho_0 V_0 C_{ijrs}^k \delta \varepsilon_{ij}^k dV + \int \rho_0 V_0 S_{ij} \delta \eta_{ij}^k dV = t + \Delta \mathcal{R} - \int \rho_0 V_0 S_{ij} \delta \varepsilon_{ij}^{(k-1)} dV$$

(8)

where, the case $k=1$ corresponds to eqn (7) and the displacements are updated as follows:

$$t + \Delta u_{ij}^{(k)} = t + \Delta u_{ij}^{(k-1)} + \Delta \dot{u}_{ij}$$

(9)

The relation in eqn (8) is associated with the well-known modified Newton procedure. If the left hand side of eqn (8) is also updated during the solution, the method corresponds to full Newton-Raphson technique which converges faster than the modified Newton scheme. It should be also pointed out that if the externally applied loads are considered to be deformation dependent, then the integrals in eqn (3) must be calculated over the volume and area calculated in the last time in the iteration.

Using isoparametric mapping, the associated finite element working equation of a geometrically nonlinear static analysis is obtained as:

$$0_0 K_L + 0_0 K_{NL} \Delta U^{(i)} = t + \Delta R - t + \Delta F^{(i-1)}$$

(10)

where, $0_0 K_L$ is the linear strain incremental stiffness matrix, $0_0 K_{NL}$ is the nonlinear strain (geometrical or initial stress) incremental stiffness matrix, $t + \Delta R$ is the vector of externally applied nodal loads at time $t + \Delta t$, $t + \Delta F$ is the vector of nodal point forces equivalent to the element stresses at time $t$ which is also employed corresponding to time $t + \Delta t$ and iteration (i-1) and $\Delta U^{(i-1)}$ is the vector of increments in nodal point displacements in iteration $i$, $t + \Delta U^{(0)} = t + \Delta U^{(0)} + \Delta U^{(0)}$.

These matrices for a two-dimensional axisymmetric 4-noded element are given as:

$$0_0 K_L = \int \rho_0 V_0 B^T L^0 C_i^0 B_L^0 dV$$

(11)

$$0_0 K_{NL} = \int \rho_0 V_0 B^T NL_0 S_i^0 B_{NL}^0 dV$$

(12)

$$0_0 F = \int \rho_0 V_0 B^T L^0 S_i^0 dV$$

(13)

where:

$$0_0 B_L = 0_0 B_{L0} + 0_0 B_{L1}$$

(14)

(1) In a static analysis without time effects, time is only a convenient variable, which denotes different intensities of load applications.
\[
\begin{align*}
^{t}B_{L0} &= \begin{bmatrix}
\frac{\partial N_1}{\partial^0 r} & 0 & \frac{\partial N_2}{\partial^0 r} & 0 & \frac{\partial N_3}{\partial^0 r} & 0 & \frac{\partial N_4}{\partial^0 r} & 0 \\
0 & \frac{\partial N_1}{\partial^0 z} & 0 & \frac{\partial N_2}{\partial^0 z} & 0 & \frac{\partial N_3}{\partial^0 z} & 0 & \frac{\partial N_4}{\partial^0 z} \\
\frac{\partial N_1}{\partial^0 z} & \frac{\partial N_2}{\partial^0 z} & \frac{\partial N_3}{\partial^0 z} & \frac{\partial N_4}{\partial^0 z} & \frac{\partial N_1}{\partial^0 z} & \frac{\partial N_2}{\partial^0 z} & \frac{\partial N_3}{\partial^0 z} & \frac{\partial N_4}{\partial^0 z} \\
N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0
\end{bmatrix} \\
^{t}B_{Li} &= \begin{bmatrix}
1_{11} \frac{\partial N_1}{\partial^0 r} & 1_{21} \frac{\partial N_1}{\partial^0 r} & 1_{11} \frac{\partial N_2}{\partial^0 r} & 1_{21} \frac{\partial N_2}{\partial^0 r} \\
1_{12} \frac{\partial N_1}{\partial^0 z} & 1_{22} \frac{\partial N_1}{\partial^0 z} & 1_{12} \frac{\partial N_2}{\partial^0 z} & 1_{22} \frac{\partial N_2}{\partial^0 z} \\
1_{11} \frac{\partial N_3}{\partial^0 z} + 1_{12} \frac{\partial N_3}{\partial^0 r} & 1_{21} \frac{\partial N_3}{\partial^0 z} + 1_{22} \frac{\partial N_3}{\partial^0 r} & 1_{11} \frac{\partial N_4}{\partial^0 z} + 1_{12} \frac{\partial N_4}{\partial^0 r} & 1_{21} \frac{\partial N_4}{\partial^0 z} + 1_{22} \frac{\partial N_4}{\partial^0 r} \\
1_{33} \frac{N_3}{\partial^0 r} & 0 & 1_{33} \frac{N_4}{\partial^0 r} & 0
\end{bmatrix} \\
^{t}B_{NL} &= \begin{bmatrix}
\frac{\partial N_1}{\partial^0 r} & 0 & \frac{\partial N_2}{\partial^0 r} & 0 & \frac{\partial N_3}{\partial^0 r} & 0 & \frac{\partial N_4}{\partial^0 r} & 0 \\
\frac{\partial N_1}{\partial^0 z} & 0 & \frac{\partial N_2}{\partial^0 z} & 0 & \frac{\partial N_3}{\partial^0 z} & 0 & \frac{\partial N_4}{\partial^0 z} & 0 \\
0 & \frac{\partial N_1}{\partial^0 r} & 0 & \frac{\partial N_2}{\partial^0 r} & 0 & \frac{\partial N_3}{\partial^0 r} & 0 & \frac{\partial N_4}{\partial^0 r} \\
0 & \frac{\partial N_1}{\partial^0 z} & 0 & \frac{\partial N_2}{\partial^0 z} & 0 & \frac{\partial N_3}{\partial^0 z} & 0 & \frac{\partial N_4}{\partial^0 z} \\
N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0
\end{bmatrix}
\end{align*}
\]
where:

\[ I_{11} = \sum_{k=1}^{N} \frac{\partial N_k}{\partial r} u_r^k \]

\[ I_{12} = \sum_{k=1}^{N} \frac{\partial N_k}{\partial z} u_z^k \]

\[ I_{21} = \sum_{k=1}^{N} \frac{\partial N_k}{\partial r} u_z^k \]

\[ I_{22} = \sum_{k=1}^{N} \frac{\partial N_k}{\partial z} u_r^k \]

\[ I_{33} = \sum_{k=1}^{N} \frac{N_k}{0} u_r^k \]

Subscripts 1, 2 and 3 refer to longitudinal, lateral and normal to plane axes, respectively. The engineering constants \( E, \nu \) and \( G \) denote the Young’s modulus, Poisson’s ratio and shear modulus, respectively. These parameters have been analytically expressed by many authors (see for example references 4 and 5) in terms of the engineering constants of the rubber and cords and their volume fractions in the composites. The tensor representing the constitutive equation in the working eqn (10) is in the global coordinate system \((r,z,\theta)\). Consequently, the elements of the matrix \([C]^{-1}\) in eqn (22) must be transformed from lamina coordinate system \((1,2,3)\) to global coordinate system \((r,z,\theta)\).

Most of the finite element codes for the structural analysis of axisymmetric structures consider only such transformation in \((r,z)\) plane which means that no out-of-plane transformation is adopted. However, in some parts of pneumatic tyres such as belts in radial tyres, reinforcing cords make non-orthogonal direction with global coordinate axes. Thus, an out-of-plane transformation is required to take the effect of cord angle into account in an axisymmetric finite element model.

The novel aspect of the present work is to modify the source code of the NSTAR module of COSMOS/M software package to take the effect of the cord angle...
into consideration. Consider a single ply which its cord make an angle $\theta$ with circumferential direction (Figure 1). The transformations of tensorial strain and stress tensors between the coordinate system (1,2,3) which is the lamina coordinates to coordinate system (1',2',3') which is the elemental local coordinates are given as[3]:

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

(23)

and

$$\begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \tau_{12}' \\ \tau_{12}' \end{bmatrix} = [T] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

(24)

where,

$$[T] = \begin{bmatrix} a_{11}a_{11} & a_{12}a_{21} & a_{12}a_{21} + a_{22}a_{11} & a_{31}a_{31} \\ a_{12}a_{12} & a_{22}a_{22} & a_{12}a_{22} + a_{22}a_{12} & a_{32}a_{32} \\ a_{11}a_{12} & a_{21}a_{22} & a_{11}a_{22} + a_{21}a_{12} & a_{31}a_{32} \\ a_{13}a_{13} & a_{23}a_{23} & a_{13}a_{23} + a_{23}a_{13} & a_{33}a_{33} \end{bmatrix}$$

(25)

The components of the matrix $[T]$ can be calculated by the following relations:

- $\begin{bmatrix} a_{11} = \cos \theta \\ a_{12} = 0 \\ a_{13} = \sin \theta \end{bmatrix}$
- $\begin{bmatrix} a_{21} = 0 \\ a_{22} = 1 \\ a_{23} = 0 \end{bmatrix}$
- $\begin{bmatrix} a_{31} = -\sin \theta \\ a_{32} = 0 \\ a_{33} = \cos \theta \end{bmatrix}$

thus:

$$[T] = \begin{bmatrix} \cos^2 \theta & 0 & 0 & -\sin^2 \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & 0 \\ \sin^2 \theta & 0 & 0 & \cos^2 \theta \end{bmatrix}$$

(26)

Substituting eqn (21) into eqn (23) yields:

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \gamma_{12}' \\ \varepsilon_3' \end{bmatrix} = [T] \begin{bmatrix} C \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \varepsilon_3 \end{bmatrix}$$

(27)

Using eqn (24), the above equation can now be written as:

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \gamma_{12}' \\ \varepsilon_3' \end{bmatrix} = [T]_2 \begin{bmatrix} C \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \sigma_3 \end{bmatrix}$$

(28)

where:

$$[T]_2 = \begin{bmatrix} \cos^2 \theta & 0 & 0 & -\sin^2 \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\cos \theta & 0 \\ \sin^2 \theta & 0 & 0 & \cos^2 \theta \end{bmatrix}$$

(29)

$$[T]^{-1} = \begin{bmatrix} \cos^2 \theta & 0 & 0 & -\cos^2 \theta \\ \frac{1}{2\cos^2 \theta -1} & 0 & 0 & \frac{-\cos^2 \theta-1}{2\cos^2 \theta-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \cos \theta \\ \frac{\cos^2 \theta-1}{2\cos^2 \theta-1} & 0 & 0 & \frac{\cos^2 \theta}{2\cos^2 \theta-1} \end{bmatrix}$$

(30)
A further transformation from coordinates \((1',2',3')\) to global coordinate system \((r,z,\theta)\) should also be carried out. This is automatically accomplished by the finite element code NSTAR.

Based on the above-described scheme, subroutine UM2D01 in the source code NSUM.F has been modified[1]. This code is then complied and by linking to the supplied object modules and library files, a modified version of NSTAR module has been created. This executable code is then used for the analysis of a steel belted radial tyre under axisymmetric loading.

**NUMERICAL RESULTS AND DISCUSSION**

In order to evaluate the applicability of the developed model it is used to analyze a 175/70R14 steel-belted tyre under inflation pressure load. This tyre has been comprised of 11 different parts. The cord angles for plies and cap-ply are 90° and 0° with respect to circumferential direction, respectively. Therefore, the axis of orthotropy in these parts coincides with the axis of symmetry. Three different belt angles \((15°, 18°\text{ and } 21°)\) have been selected for the analyses. In addition, this structure has also been analyzed without cap ply to take the effect of the cap ply stiffness on the deformed shape of the tyre into account.

The 4-noded isoparametric elements with total Lagrangian large deformation formulation was used for the modeling of the tyre. Taking advantage of tyre symmetry, only one-half of the tyre was modeled. Figure 2 shows the finite element mesh of the tyre. The total number of nodes and elements used in this mesh are 243 and 214, respectively. Having tried several mesh densities the selected mesh is found to be convergent and accurate for this problem.

In order to model the elastic behaviour of the tyre, two different material models have been used. Linear isotropic elastic model was used for the rubbery parts

**Figure 2.** Finite element mesh of the tyre.

**Figure 3.** Applied pressure direction and imposed boundary conditions.

**Figure 4.** Build-up of crown displacement with internal pressure as a function of belt angle.

**Figure 5.** Deformed shapes of the tyre under pressure load (belt angle=18°).
such as tread, under tread, sidewall, etc and linear orthotropic model was chosen for the description of the behaviour of cord-rubber composites. The elastic constants of each component are recorded in Table 1. There are various empirical models for the calculation of the elastic constants of an orthotropic cord-rubber lamina. However, it was shown that all of these models give nearly identical results [4]. The author has successfully used Clark’s theory [5] to compute the Young’s modulus and Poisson’s ratio for a cord-rubber composite in a pneumatic tyre [6]. In the present work the same approach was also adopted.

The inflation pressure is set to 0.294 MPa (3 kg/cm²) which has been applied in 10 equal stages. Figure 3 shows the direction of the applied pressure in conjunction with the imposed boundary conditions.

Displacements of the crown of the tyre as a function of inflation pressure for three belt angles are plotted in Figure 4. As the belt angle increases, the circumferential stiffness becomes more compliant and thus the radial growth increases. The deformed predicted shape at 0.294 MPa inflation pressure with belt angle 18° is shown in Figure 5. The calculated increase in tyre outer diameter (OD) for belt angle equal to 18° is 1.37 mm. It is experimentally found that the growths in OD of steel belted radial tyres with belt angles equal to above value are usually between 1 to 1.5 mm. Taking into account the effect of shrinkage upon opening of the mould and thus selection of uninflated shape as initial configuration, the accuracy of the developed model has been verified.

In order to investigate the effect of the cap ply stiffness on the deformed configuration, the finite element analysis was also performed with cap ply stiffness equal to the rubber modulus. It is found that in this case tyre shows a 12% more growth in radial direction at crown. This is obviously due to the reduction in the stiffness of the under tread area of the tyre.

### CONCLUSION

Application of the axisymmetric finite element method is still the basic stage during the stress-strain analysis of pneumatic tyres. However, due to the variation in cord angle in cord-rubber composites used in tyre body, the axis of orthotropy does not coincide with the axis of symmetry and thus the conventional axisymmetric formulation is not adequate to cope with this circumstance. In the present research the source code of a commercial nonlinear finite element code has been modified by the use of an in-house developed complementary code to take the effect of cord angle on the material properties of the tyre carcass constituents such as belt, ply, etc. The modified code is then used for the simulation of a steel-belted radial tyre under internal pressure load. It is shown here that the value of the belt configuration.
angle has great influence on the deformed shape and especially at the crown of the tyre. Also the validity and applicability of the developed model have been verified by the comparison between computed values of the displacement at crown with experimentally measured values of the tyre outer diameter.

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