A Numerical Study on the Non-linear Finite Element Analysis of a Tyre under Axisymmetric Loading

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ABSTRACT

This research work is devoted to the numerical study of the non-linear finite element analysis of pneumatic tyres. The purpose of the present work is to find appropriate numerical schemes for the non-linear stress-strain analysis of such complex structures. Different numerical options including full and modified Newton-Raphson methods, order of interpolation functions, incompatible modes and type of non-linear formulations (total Lagrangian and up-dated Lagrangian) have been considered during this parametric study. These methods have been applied to the finite element analysis of a homogeneous tyre under axisymmetrical pressure loading. The final decision on the selection of the best approach has been made by the comparison of the numerical results with available experimentally measured data. It is shown that the combination of incompatible modes in case of lower order elements or total Lagrangian formulation with higher order elements as well as the use of full Newton-Raphson method provide the best accurate results.

Key Words: finite element, tyre, non-linear, numerical study, Lagrangian method.

INTRODUCTION

It is now quite general to use the non-linear finite element analysis in a non-linear design development cycle to make appropriate design decisions and also to assess the product performance. Numerous researchers have extensively applied this method for non-linears under various static and dynamic loads [1-8]. Despite the simple linear analysis in which only a few options are available, most of the advanced non-linear finite element codes offer a broad range of numerical schemes.
All of these schemes are normally valid for non-linear analyses. For example, both total Lagrangian and updated Lagrangian formulations can be successfully applied to a specific problem.

However, choosing the best method within the available options needs a thorough understanding of the nature of the non-linear finite element formulation and also to make comparisons of the results with actual data.

In the present research, the effects of the various numerical schemes and non-linear formulations have been studied. This work consists of the analysis of a pneumatic tyre under pressure loading. Different parameters including the use of full and modified Newton-Raphson methods, incompatible modes, order of interpolation functions and total Lagrangian versus updated Lagrangian have been taken into account during this work. This methodology has been applied to the stress-strain analysis of homogeneous polyurethane tyre under pressure loading. The NSTAR code has also been used for this purpose. Having compared the results of the analysis with available experimental data, the best suitable method is found. Therefore, this method can be applied to other similar systems to obtain highly reliable and accurate results.

In the following sections we first describe the mathematical background of the problem. This includes the basic relations for the non-linear finite element formulation of a general body under external loads and also introducing of the appropriate measures for stress and strain in a large deformation analysis. Then, the finite element model of the tyre and numerical results are presented. Finally the conclusions are drawn and the best numerical schemes for such problems are presented.

**Theoretical Background**

Consider the large deformation of a body under load in a stationary Cartesian coordinate system from an initial configuration to the deformed state. In a non-linear finite element analysis the equilibrium of the body must be established in the current (deformed) configuration. Assuming that the loads are applied as a function of time, the aim is to predict the equilibrium of the body at discrete time points 0, Δt, 2Δt, 3Δt, ... where Δt is an increment in time. This means that an incremental formulation is employed. In the solution strategy it is assumed that solutions for all time steps from time zero to time t has been obtained and the solution correspon-
Where, \( t+\Delta t \mathbf{f}_i^B \) and \( t+\Delta t \mathbf{f}_i^S \) are the components of the externally applied body and surface force vectors referred to time \( t+\Delta t \), respectively. In a static analysis without time effects, time is only a convenient variable, which denotes different intensities of load applications. However, when time is meaningful such as dynamic analysis or static analysis with creep effect, the time variable is an actual parameter to be properly included in the mathematical formulation of the problem.

The configuration of the body changes continuously in a large deformation analysis, and thus, appropriate stress and strain measures as well as constitutive relations should be used. This is because the internal virtual work in eqn (1) should be expressed in terms of an integral over a known volume and to be able decompose stresses and strains incrementally in an effective manner. The stress used for the non-linear structural finite element analysis is 2nd Piola-Kirchhoff stress. At time \( t \) this stress is referred to configuration of the body at time 0 and defined as:

\[
\mathbf{S}_0 = \frac{1}{\rho} \left( \mathbf{0} + \mathbf{x}_{i,m} \right) \mathbf{\tau}_{mn} \mathbf{x}_{j,n}
\]  

(4)

The strain tensor used with the 2nd Piola-Kirchhoff stress tensor is the Green-Lagrange strain tensor defined as:

\[
\mathbf{e}_0 = \frac{1}{2} \left( \mathbf{u}_{i,j} + \mathbf{u}_{j,i} \right)
\]  

(5)

Where, \( S \) is the stress tensor, \( \mathbf{e} \) is the strain tensor, \( \rho \) is the density of the material and \( \mathbf{x}_{i,m} \) is the element (i,m) of the gradient deformation tensor.

\[
\mathbf{x} = (\mathbf{V} \mathbf{X}^T)^T
\]

In the above equations a “comma” denotes differentiation with respect to the coordinate following e.g., \( \mathbf{x}_{i,m} = \partial^\mathbf{x_i} / \partial \mathbf{x_m} \). The left subscript indicates the configuration in which this coordinate is measured and the left superscript indicates in which one of the configurations the quantity occurs. Both the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors are symmetric and objective tensors, which means that rigid body motions of the material do not alter their components. Using these new stress and strain measures, the basic eqn (1) can now be expressed by:

\[
\int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV = (t+\Delta t) \mathcal{R}
\]  

(6)

The above formulation is generally called total Lagrangian (T.L.) formulation in which all static and kinematic variables are referred to the initial configuration at time 0. There is also another formulation called updated Lagrangian (U.L.) in which all static and kinematic variables are referred to the known configuration at time \( t \). The corresponding equation to this formulation is given as:

\[
\int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV = (t+\Delta t) \mathcal{R}
\]  

(7)

The approximate solutions to above equations are obtained by linearizing of eqns (6) and (7), which are given for T.L. and U.L. formulations, respectively, by:

\[
\int_{V} \mathbf{C}_{i,jrs}^0 \mathbf{e}_0^j \mathbf{\eta}_0^j \, dV + \int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV = (t+\Delta t) \mathcal{R} - \int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV
\]  

(8)

\[
\int_{V} \mathbf{C}_{i,jrs} \mathbf{e}_0^j \mathbf{\eta}_0^j \, dV + \int_{V} \mathbf{\tau}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV = (t+\Delta t) \mathcal{R} - \int_{V} \mathbf{\tau}_0^t \mathbf{\delta}_0^t \mathbf{e}_0^t \, dV
\]  

(9)

Where, \( \mathbf{C}_{i,jrs}^0 \) and \( \mathbf{C}_{i,jrs} \) are the incremental material property tensors at time \( t \) referred to the configurations at times 0 and \( t \), respectively; \( \mathbf{e}_0^j \) and \( \mathbf{e}_0^j \) are the linear incremental strain tensors and \( \mathbf{\eta}_0^j \) and \( \mathbf{\eta}_0^j \) are the non-linear incremental strain tensors which are referred to the configuration at time 0 and \( t \), respectively. It should also be noted that \( \mathbf{S}_0^t \) and \( \mathbf{\tau}_0^t \) are the known 2nd Piola-Kirchhoff and Cauchy stresses at time \( t \).

The right hand side of eqns (8) and (9) represent out-of-balance virtual work, which should be continuously reduced within a certain convergence tolerance by performing iteration during the solution of this equation. The iterative format of eqn (8) which is in the (T.L.) framework, for \( k=1,2,\ldots \) is:

\[
\int_{V} \mathbf{C}_{i,jrs}^k \mathbf{\eta}_0^j \, dV + \int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{\eta}_0^{(k)} \, dV = (t+\Delta t) \mathcal{R} - \int_{V} \mathbf{S}_0^t \mathbf{\delta}_0^t \mathbf{\eta}_0^{(k-1)} \, dV
\]  

(10)

and for (U.L.) expressed by:
The relations in eqns (10) and (11) are associated with the well-known modified Newton-Raphson (MNR) procedure. If the left-hand side of these equations are updated during the solution, the method corresponds to full Newton-Raphson (NR) technique, which converges faster than the modified Newton-Raphson scheme. It should be also pointed out that if the externally applied loads are considered to be deformation dependent which is the case of tyre inflation problem, then the integrals in eqn (3) must be calculated over the volume and area calculated in the last time in the iteration.

Using isoparametric mapping, the associated finite element working equation of a geometrically non-linear static analysis are obtained for T.L. and U.L., respectively, as:

\[
\begin{align*}
\sum_{j=1}^{m} \left( K_{L}^{(e)} + K_{NL}^{(e)} \right) U^{(i)} & = \mathbf{R}_{L}^{(i-1)} + \mathbf{F}^{(i-1)} \\
\sum_{j=1}^{m} \left( K_{L}^{(e)} + K_{NL}^{(e)} \right) U^{(i)} & = \mathbf{R}_{L}^{(i-1)} + \mathbf{F}^{(i-1)} 
\end{align*}
\]

Where, \( K_{L}^{(e)} \) and \( K_{NL}^{(e)} \) are the linear strain incremental stiffness matrices, \( K_{L}^{(e)} \) and \( K_{NL}^{(e)} \) are the non-linear strain (geometrical or initial stress) incremental stiffness matrices, \( R_{L}^{(i-1)} \) is the vector of externally applied nodal loads at time \( t+\Delta t \), \( F^{(i-1)} \) and \( F^{(i)} \) are the vectors of nodal point forces equivalent to the element stresses at time \( t \) which is also employed corresponding to time \( t+\Delta t \) and iteration \( (i-1) \) and \( \Delta U^{(i-1)} \) is the vector of increments in nodal point displacements in iteration \( i \):

\[
\begin{align*}
\mathbf{R}_{L}^{(i+1)} & = \mathbf{R}_{L}^{(i)} + \Delta \mathbf{R}^{(i-1)} \\
\mathbf{U}^{(i+1)} & = \mathbf{U}^{(i)} + \Delta \mathbf{U}^{(i)} 
\end{align*}
\]

The details of the associated working equations to these stiffness matrices and load vectors are written in reference [11] and thus it is not repeated here. In the present study we have used two isoparametric elements in two-dimensional axisymmetric framework. These are 4-noded bilinear and 8-noded serendipity biquadratic elements. The interpolations (shape) functions for these elements are given in reference [9]. These functions are used to approximate the displacement vector \( \{u\} \) over a typical element \( (e) \) in axisymmetric coordinate system \((r,z)\) using these well-known expressions:

\[
\begin{align*}
u_{r}(r,z) & = \sum_{j=1}^{m} u_{rj}^{(e)} \psi_{j}^{(e)}(\xi,\eta) \\
u_{z}(r,z) & = \sum_{j=1}^{m} u_{zj}^{(e)} \psi_{j}^{(e)}(\xi,\eta) 
\end{align*}
\]

Where, \( m \) is the number of nodes per elements and \( u_{r} \) and \( u_{z} \) are the components of the displacement vector \( \{u\} \) in \( r \) and \( z \) directions, respectively. However, in some cases in order to improve the behaviour of the 4-noded isoparametric elements, extra terms are added to the above displacement equations, which is known as “incompatible modes”. In this case the displacement interpolations used are:

\[
\begin{align*}
u_{r}(r,z) & = \sum_{j=1}^{4} u_{rj}^{(e)} \psi_{j}^{(e)}(\xi,\eta) + \alpha_{1}(1-\xi^{2}) + \alpha_{2}(1-\eta^{2}) \\
u_{z}(r,z) & = \sum_{j=1}^{4} u_{zj}^{(e)} \psi_{j}^{(e)}(\xi,\eta) + \beta_{1}(1-\xi^{2}) + \beta_{2}(1-\eta^{2}) 
\end{align*}
\]

It should be noted that using the interpolation relations in eqns (17) and (18), a stiffness matrix of order 12 is obtained (rather than of order 8 in case of using normal relations 15 and 16 with \( m=4 \)). However, four parameters \( \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \), are not associated with any nodes in the element. Thus, to reduce the stiffness matrix to order 8, additional degrees of freedom (corresponding to these four extra parameters) are eliminated using static condensation method. This method is the simple application of the well-known Gauss elimination technique, which is extensively used for the solution of the linear system of equations [9,11]. Having computed the elemental stiffness matrix and load vectors, they are assembled over common degrees of freedom and solved using either a modified Newton-Raphson or Newton-Raphson technique.

**Finite Element Model**

A homogeneous 70-15 pneumatic tyre made of polyurethane has been selected for the present analysis. This tyre had been previously analyzed under both inflation and contact loading [3]. In order to study the
effects of the different numerical schemes in the tyre finite element analysis, this tyre has been re-analyzed under inflation loading based on the use of different above mentioned methods. To take into account the effect of mesh refinement and element type, three mesh patterns with two element types (4-noded bi-linear and 8-noded bi-quadratic/serendipity) have been selected which are labeled as mesh type I, II and III, respectively. These mesh configurations are shown in Figures 2-4. A linear elastic material model is also used to express the stress-strain behaviour of the rubber. The elastic properties of the present polyurethane tyre are 26.2 MPa for Young’s modulus and 0.48 for Poisson’s ratio, respectively [3]. Tyre inflation pressure is also set to 0.1655 MPa (24 psi), which is applied, in eight equal increments.

Numerical Results and Discussion

A total of 36 simulations have been performed for the present model using NSTAR Ver. 1.75 [13] software which, has been already used for the analysis of pneumatic tyres under different loading [4,8]. Figure 5 shows boundary conditions and applied load used for all analyses. In these simulations, effects of the element order (4-noded and 8-noded), type of interpolation functions (isoparametric and incompatible modes), large deformation formulation (total Lagrangian and updated Lagrangian) and finally technique for the solution of the tyre equations (modified Newton-Raphson and full Newton-Raphson) are studied. To determine the extent of validity of each of the selected numerical scheme, the results of the simulations are compared with the experimentally measured displacement at the crown of the tyre [3]. This is the most critical point in a tyre, in which its deformation is checked by tyre designers at the every stage of the development of a non-linear layout.

Early analyses apparently showed that the differences between results obtained by the modified Newton-Raphson (MNR) method and those obtained by Newton-Raphson (NR) technique are negligible and thus NR is used as the method of choice during the rest
of simulations. (Figure 6). However, as it is expected obtaining convergent results with MNR method requires more iterations and in some cases smaller time steps should be adopted in the analysis. Build-up of crown displacement with the inflation pressure for three mesh types I, II and III and two NPEs (number of nodes per element) are shown in Figures 7-12, respectively. In each graph, the crown displacement has been plotted against inflation pressure with their corresponding integration (QM6 and Full) and non-linear (U.L. and T.L.) methods. Here, QM6 and Full integration methods (from now on referred to Full) refer to the use of incompatible mode and simple Gauss integration technique and U.L. and T.L. refer to Update Lagrangian and Total Lagrangian non-linear formulations, respectively [13]. As it is mentioned above, QM6 method was only applied for 4-noded (NPE=4) case.

Figures 7 and 8 the show displacement-pressure curve for mesh type I and NPE equal to 4 and 8, respectively. As it can be seen, with a constant mesh density, results obtained by QM6 method and 4-noded elements are almost identical with those obtained by Full method and 8-noded elements. Since in QM6 elements additional terms are added to the displacement interpolation relation, therefore the behaviour of element is similar to higher order elements. Displacements predicted by the T.L. method is always lower than those computed by U.L. method. Comparing the computed results with experimentally measured data shows that at lower pressure very excellent confirmations exist for higher order elements. However, when pressure increases, discrepancies between actual and evaluated data become more prominent. This is due to the highly non-linear behavior of elastomeric materials even at relatively low loads. It is also apparent from the results that the T.L. method gives more accurate results than U.L. technique.
The results for a more refined mesh pattern are shown in Figures 9–10, respectively. Similar behaviour is also observed in this case. However, comparison of the results with experimental data shows that the differences between results of the 4-noded finite element model with 8-noded model is less prominent, which is obviously due to the improvement of the model achieved by the mesh refinement. It is anticipated that with mesh refinement the difference between the T.L. and U.L. method is vanished. But, in this case these two non-linear methods do not converge and both schemes should be used. Thus comparison of the results with actual data would be the final decision on choosing the appropriate approach in such problems. However, as it is mentioned above, T.L. method gives more accurate results than U.L. method. From mesh refinement point of view, it can be also observed that results obtained by QM6 scheme is more stable than those achieved by Full method.

Figures 11 and 12 show the same displacement pressure curves for our final refined mesh. As it can be seen, by comparison of the results with both actual data and results shown in previous figures, the trend of the displacement-pressure behaviour is to be more convergent which is a quite obvious outcome since by refining the finite element mesh more accurate and convergent results are generally obtained.

CONCLUSION

In the present research the effects of various options, which are generally available during the non-linear analysis of structures, have been extensively studied for a homogeneous pneumatic tyre under axisymmetric loading. This will enable tyre analysts to find the best appropriate method for the stress-strain analysis of the primary designs. Based on the results obtained in this work and also comparison with experimentally measured data, the following outcomes can be summarized:

1-Due to the superior capabilities in obtaining convergent results, the full Newton-Raphson method is found to be the best method for the solution of the non-linear equations.

2-For relatively refined meshes, computed non-linear load-displacement behaviour is nearly independent of number of nodes per element.

3-Use of incompatible modes requires more iterations to obtain convergent results than the full scheme.

4-A combination of total Lagrangian method with QM6 (incompatible modes) gives the best results.

5-The calculated error levels between experimental data and computed result are strongly dependent to mesh density, type of element, integration order and non-linear formulations. Here we have found that the error range is between 1.5% up to 30% at
the crown of the tyre.

It should be noted that although the present approach was restricted to the simulation of a homogeneous tyre, the methodology adopted could be readily extended to other non-linear finite element applications to identify the best numerical scheme for any particular problem.

REFERENCES