FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let \( R \) be a commutative Noetherian ring. In this paper we consider some relations between filter regular sequence, regular sequence and system of parameters over \( R \)-modules. Also we obtain some new results about cofinitness and cominimaxness of local cohomology modules.

1. Introduction

Throughout this paper, let \( R \) denote a commutative Noetherian ring (with identity) and \( I \) an ideal of \( R \). For an \( R \)-module \( M \), the \( i \)th local cohomology module of \( M \) with respect to \( I \) is defined as

\[
H^i_I(M) = \lim_{n \to \infty} \operatorname{Ext}_R^i(R/I^n, M).
\]

We refer the reader to [5] or [3] for more details about local cohomology. The concept of filter regular sequence plays an important role in this paper. We say that a sequence \( x_1, \ldots, x_n \) of elements of \( I \), is an \( I \)-filter regular sequence on \( M \), if

\[
\operatorname{Supp}_R \left( \frac{(x_1, \ldots, x_{i-1})M}{(x_1, \ldots, x_{i-1})M} : x_i \right) \subseteq V(I),
\]

for all \( i = 1, \ldots, n \). Also, we say that an element \( x \in I \) is an \( I \)-filter regular sequence on \( M \) if \( \operatorname{Supp}_R(0 :_M x) \subseteq V(I) \). The concept of an \( I \)-filter regular sequence on \( M \) is a generalization of the concept of a filter

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regular sequence which has been studied in [18]. Both concepts coincide if $I$ is an $m$-primary ideal of a local ring with maximal ideal $m$. In 1969, A. Grothendieck conjectured that if $I$ is an ideal of $R$ and $M$ is a finitely generated $R$-module, then the $R$-modules $\text{Hom}_R(R/I, H^i_I(M))$ are finitely generated for all $i \geq 0$. R. Hartshorne has provided a counterexample to this conjecture in [6]. Also he defined a module $T$ to be $I$-cofinite if $\text{Supp} T \subseteq V(I)$ and $\text{Ext}^i_R(R/I, T)$ is finitely generated for each $i \geq 0$ and he asked the following question.

For which rings $R$ and ideals $I$ are the modules $H^i_I(M)$ $I$-cofinite for all $i$ and all finitely generated modules $M$?

Hartshorne proved that if $I$ is an ideal of the complete regular local ring $R$ and $M$ a finitely generated $R$-module, then $H^i_I(M)$ is $I$-cofinite in two following cases:
(i) $I$ is principal ideal, (see [6], Corollary 6.3),
(ii) $I$ is prime ideal with dim $R/I = 1$, (see [6], Corollary 7.7).
This subject was studied by several authors afterwards, (see [4], [11], [9], [19], [1] and [10]).

Some important results of this paper are as follows:

**Theorem 1.1.** Let $(R, m)$ be a Noetherian local ring and $M \neq 0$ be a finitely generated $R$-module of dimension $d \geq 1$. Let $x_1, \ldots, x_d \in m$ be an $m$-filter regular sequence for $M$. Then the following statements are holds:

1. $x_1, \ldots, x_d$ is a system of parameters for $M$.
2. For each $1 \leq i \leq d$, the $R$-module $H^i_m(M)$ is $(x_1, \ldots, x_i)$-cofinite.

**Theorem 1.2.** Let $(R, m)$ be a Noetherian local ring and $I$ be an ideal of $R$. Then for every finitely generated $R$-module $M \neq 0$ of dimension $d$, the following statements are equivalent:

1. $H^d_m(M)$ is $I$-cofinite.
2. $H^d_m(M) \cong H^d(I)(M)$.

**Theorem 1.3.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M \neq 0$ be a finitely generated $R$-module such that $\dim \frac{M}{IM} \leq 1$. If $t \geq 1$ and $x_1, \ldots, x_t \in I$ is an $I$-filter regular sequence for $M$, then for each $0 \leq i \leq t - 1$, the $R$-module $H^i_I(M)$ is $(x_1, \ldots, x_t)$-cofinite and $\text{Hom}_R\left(\frac{R}{(x_1, \ldots, x_t)}, H^i_I(M)\right)$ is finitely generated.
For each $R$-module $L$, we denote by $\text{Ass}_RL$ the set $\{p \in \text{Ass}_RL : \dim R/p = \dim L\}$. Also, for any ideal $b$ of $R$, the radical of $b$, denoted by $\text{Rad}(b)$, is defined to be the set $\{x \in R : x^n \in b \text{ for some } n \in \mathbb{N}\}$ and we denote $\{p \in \text{Spec}(R) : p \supseteq b\}$ by $V(b)$. Finally, for each $R$-module $L$, we denote by $\text{mAss}_RL$, the minimal elements of $\text{Ass}_RL$. For any unexplained notation and terminology we refer the reader to [3] and [12].

2. Main results

Theorem 2.1. Let $(R, m)$ be a Noetherian local ring and $M \neq 0$ be a finitely generated $R$-module of dimension $d \geq 1$. Let $x_1, \ldots, x_d \in m$ be an $m$-filter regular sequence for $M$. Then

1. $x_1, \ldots, x_d$ is a system of parameters for $M$.
2. For each $1 \leq i \leq d$, the $R$-module $H^i_m(M)$ is $(x_1, \ldots, x_i)$-cofinite.

Proof. (1). By definition $x_i \notin \cup_{p \in \text{Ass}(\frac{R}{(x_1, \ldots, x_{i-1})}\setminus\{m\})} P$ for each $1 \leq i \leq d$, and so $x_i \notin \cup_{p \in \text{Ass}_R(\frac{R}{(x_1, \ldots, x_{i-1})})} P$. Therefore $x_1, \ldots, x_d$ is a system of parameters for $M$.

(2). By [8, Proposition 1.2], $H^j_{(x_1, \ldots, x_i)}(M) \cong H^j_m(M)$ for each $0 \leq j \leq i - 1$ and $\dim \text{Supp} H^j_{(x_1, \ldots, x_i)}(M) \leq 0$. Hence by [1, Theorem 2.6], the $R$-module $H^j_{(x_1, \ldots, x_i)}(M)$ is $(x_1, \ldots, x_i)$-cofinite. Also for $j > i$, $H^j_{(x_1, \ldots, x_i)}(M) = 0$. Thus by [15, Proposition 3.11], the $R$-module $H^i_{(x_1, \ldots, x_i)}(M)$ is also $(x_1, \ldots, x_i)$-cofinite. Since $H^{i-1}_{(x_1, \ldots, x_i)}(M)$ is Artinian, it follows from Grothendick vanishing theorem [3, Proposition 6.1], $H^i_{Rx_{i+1}}(H^{i-1}_{(x_1, \ldots, x_i)}(M)) = 0$. By [17], there exists an exact sequence as follows $0 \rightarrow H^i_{Rx_{i+1}}(H^{i-1}_{(x_1, \ldots, x_i)}(M)) \rightarrow H^i_{(x_1, \ldots, x_{i+1})}(M) \rightarrow H^i_{Rx_{i+1}}(H^i_{(x_1, \ldots, x_i)}(M)) \rightarrow 0$. Note that this exact sequence shows

$H^i_{(x_1, \ldots, x_{i+1})}(M) \cong H^i_{Rx_{i+1}}(H^i_{(x_1, \ldots, x_i)}(M))$.

Also by [9], we have

$H^i_{(x_1, \ldots, x_{i+1})}(M) \cong H^i_m(M)$.

Therefore

$H^i_m(M) \cong H^0_{Rx_{i+1}}(H^i_{(x_1, \ldots, x_i)}(M))$

and there exists an exact sequence as $0 \rightarrow H^i_m(M) \rightarrow H^i_{(x_1, \ldots, x_i)}(M)$. Since $\text{Hom}_R\left(\frac{R}{(x_1, \ldots, x_i)}, H^i_{(x_1, \ldots, x_i)}(M)\right)$ is finitely generated (because $H^i_{(x_1, \ldots, x_i)}(M)$ is $(x_1, \ldots, x_i)$-cofinite), it follows that the $R$-module
\[ \text{Hom}_R \left( \frac{R}{(x_1,\ldots,x_i)}, H^i_\mathfrak{m}(M) \right) \] is also finitely generated. Now, by [16, Theorem 1.6] and by Artinianess of \( H^i_\mathfrak{m}(M) \), we conclude that \( H^i_\mathfrak{m}(M) \) is \((x_1,\ldots,x_i)\)-cofinite. \( \square \)

**Theorem 2.2.** Let \((R,\mathfrak{m})\) be a complete Noetherian local ring and \( M \neq 0 \) be a finitely generated \( R \)-module of dimension \( d \geq 1 \). Let \( P \in \text{Ass} M \) be such that \( \dim \frac{R}{P} = t \geq 1 \). Then for any \( \mathfrak{m} \)-filter regular sequence for \( M \) such as \( x_1,\ldots,x_t \in \mathfrak{m} \), \( \text{Rad}(P + (x_1,\ldots,x_t)) = \mathfrak{m} \). In particular \( x_1,\ldots,x_t \) is a system of parameters for \( \frac{R}{P} \).

**Proof.** By Cohen’s theorem every complete Noetherian ring is a homomorphic image of a Gorenstein local ring. Then by [2], we have

\[ \{ q \in \text{Att}_R H^i_\mathfrak{m}(M) \mid \dim \frac{R}{q} = t \} = \{ q \in \text{Ass} M \mid \dim \frac{R}{q} = t \}. \]

Since \( P \in \text{Ass} M \) and \( \dim \frac{R}{P} = t \), it follows that \( P \in \text{Att} H^i_\mathfrak{m}(M) \).

By the previous Theorem, the \( R \)-module \( H^i_\mathfrak{m}(M) \) is \((x_1,\ldots,x_t)\)-cofinite and so by [16, Theorem 1.6], \( \text{Rad}(P + (x_1,\ldots,x_t)) = \mathfrak{m} \). \( \square \)

**Theorem 2.3.** Let \((R,\mathfrak{m})\) be a Noetherian local ring and \( I \) be an ideal of \( R \). Then for every finitely generated \( R \)-module \( M \neq 0 \) of dimension \( d \), the following statements are equivalent.

1. \( H^d_\mathfrak{m}(M) \) is \( I \)-cofinite.
2. \( H^d_\mathfrak{m}(M) \cong H^d_I(M) \).

**Proof.**

1 \( \to \) 2 Let \( H^d_\mathfrak{m}(M) \) be \( I \)-cofinite module. Then \( H^d_\mathfrak{m}(M) \otimes_R \hat{R} \) is also \( I\hat{R} \)-cofinite. Hence by [16, Theorem 1.6], for each \( P \in \text{Att}_R (H^d_\mathfrak{mR}(\hat{M})) = \text{Assh}_R(\hat{M}), \text{Rad}(I\hat{R} + P) = \mathfrak{m}\hat{R} \) and so \( H^d_{I\hat{R}}(\frac{\hat{R}}{P}) \neq 0 \). Therefore \( H^d_{I\hat{R}}(\hat{R}) \otimes_{\hat{R}} \frac{\hat{R}}{P} \neq 0 \) and \( P \in \text{Att}_R H^d_{I\hat{R}}(\hat{R}) \). Consequently \( \text{Att}_{\hat{R}} H^d_{\mathfrak{mR}}(\hat{R}) \subseteq \text{Att}_{\hat{R}} H^d_{I\hat{R}}(\hat{R}) \subseteq \text{Att} H^d_{\mathfrak{mR}}(\hat{R}) \) and so \( \text{Att}_{\hat{R}} (H^d_{\mathfrak{mR}}(\hat{R})) = \text{Att}_{\hat{R}} (H^d_{I\hat{R}}(\hat{R})) \). Now by [7], \( H^d_{\mathfrak{mR}}(\hat{R}) \cong H^d_{I\hat{R}}(\hat{R}) \). Hence we have the following:

\[ H^d_m(R) \cong H^d_{\mathfrak{mR}}(\hat{R}) \cong H^d_{I\hat{R}}(\hat{R}) \cong H^d_I(R) \]

(2 \( \to \) 1). By [15], \( H^d_I(M) \) is \( I \)-cofinite. Since \( H^d_I(M) \cong H^d_m(M) \), it follows that \( H^d_m(M) \) is \( I \)-cofinite. \( \square \)

**Corollary 2.4.** Let \((R,\mathfrak{m})\) be a Noetherian local ring of dimension \( d \) and \( I \) be an ideal of \( R \) such that \( H^d_m(R) \) is \( I \)-cofinite. Then \( \text{ara}(I) = d \).
Proof. The module $H_d^m(R)$ is $I$-cofinite, hence $H_d^m(R) \cong H_d^m(R) \neq 0$ and so ara($I$) $\geq$ cd($I, R$) $= d$. On the other hand by [14, Corollary 2.8], ara($I$) $\leq d$. \hfill $\Box$

**Definition 2.5.** Let $I$ be an ideal of $R$. The arithmetic rank of $I$, denoted by ara($I$), is the least number of elements of $R$ required to generate an ideal which has the same radical as $I$.

**Corollary 2.6.** Let $(R, m)$ be a Noetherian local ring of dimension $d \geq 0$ and $x_1, \ldots, x_{d-1} \in m$ be such that $I = (x_1, \ldots, x_{d-1})$. Then Hom$_R \left( \frac{R}{I}, H_d^m(R) \right)$ is not finitely generated.

**Proof.** By [16, Theorem 1.6], the $R$-module Hom$_R \left( \frac{R}{I}, H_d^m(R) \right)$ is finitely generated if and only if $H_d^m(R)$ is $I$-cofinite. But in this case ara($I$) $= d$. On the other hand ara($I$) $\leq d - 1$ which is a contradiction. \hfill $\Box$

**Proposition 2.7.** Let $(R, m)$ be a complete Noetherian local ring and $M \neq 0$ be a finitely generated $R$-module. Let $N$ be submodule of $M$ such that dim $N = t \geq 1$. Then any $m$-filter regular sequence for $M$ such as $x_1, \ldots, x_t \in m$ is a system of parameters for $N$.

**Proof.** Let $m \text{Ass}_R N = \{ P_1, \ldots, P_n \}$, where $m \text{Ass}_R N$ denotes the minimal elements of $\text{Ass}_R N$. Then for each $1 \leq i \leq n$, dim $\frac{R}{P_i}$ $\leq$ dim $N = t$ and clearly dim $\frac{R}{P_i} \geq 1$. Let $j = \dim \frac{R}{P_i}$. Then $j \leq t$ and by Theorem 2.2, Rad($P_i + (x_1, \ldots, x_j)$) $= m$. Since $(x_1, \ldots, x_j) \subseteq (x_1, \ldots, x_t)$, it follows that Rad($P_i + (x_1, \ldots, x_t)$) $= m$. We claim that Rad($\cap_{i=1}^n P_i + (x_1, \ldots, x_j)$) $= m$. For this, let $Q$ be a minimal prime of $\cap_{i=1}^n P_i + (x_1, \ldots, x_t)$. Hence there exists $1 \leq j \leq n$ such that $P_j \subseteq Q$ and so $p_j + (x_1, \ldots, x_n) \subseteq Q$. Therefore $m = \text{Rad}(P_j + (x_1, \ldots, x_t)) \subseteq \text{Rad}(Q) = Q \subseteq m$ and consequently $Q = m$. But $\cap_{i=1}^n P_i = \text{Rad}(\text{Ann} N)$ shows that

$$\text{Rad}(\text{Ann} N + (x_1, \ldots, x_t)) = m$$

and so $\dim_R \frac{N}{(x_1, \ldots, x_t)N} = 0$. This completes the proof that $x_1, \ldots, x_t$ is a system of parameters for $N$. \hfill $\Box$

**Corollary 2.8.** Let $(R, m)$ be a complete Noetherian local ring, $M$ be a finitely generated $R$-module and $N$ be a submodule of $M$ which is a Cohen-Macaulay with dim $N = t$. If $x_1, \ldots, x_t \in m$ is an $m$-filter regular sequence for $M$, then $x_1, \ldots, x_t$ is a $N$-regular sequence.
Proof. By Proposition 2.7, \( x_1, \ldots, x_t \) is a system of parameters for \( N \). But \( N \) is a Maximal Cohen-Macaulay as an \( \frac{R}{Ann \, N} \)-module. Also \( x_1 + Ann \, N, \ldots, x_t + Ann \, N \) is a system of parameters for \( \frac{R}{Ann \, N} \). On the other hand every maximal Cohen-Macaulay as an \( \frac{R}{Ann \, N} \)-module is a balanced big Cohen-Macaulay as an \( R \)-module. Set \( y_i = x_i + Ann \, N \) for each \( 1 \leq i \leq t \), then \( y_1, \ldots, y_t \) is an \( N \)-regular sequence and this follows that \( x_1, \ldots, x_t \) is an \( N \)-regular sequence.

**Theorem 2.9.** Let \( R \) be a Noetherian ring, \( I \) an ideal of \( R \) and \( M \neq 0 \) be a finitely generated \( R \)-module such that \( \dim \frac{M}{IM} \leq 1 \). If \( t \geq 1 \) and \( x_1, \ldots, x_t \in I \) is an \( I \)-filter regular sequence for \( M \), then for each \( 0 \leq i \leq t - 1 \), the \( R \)-module \( H^i_t(M) \) is \((x_1, \ldots, x_t)\)-cofinite and \( \text{Hom}_R \left( \frac{R}{(x_1, \ldots, x_t)}, H^i_t(M) \right) \) is finitely generated.

Proof. For each \( 0 \leq i \leq t - 1 \), we have \( H^i_{(x_1, \ldots, x_t)}(M) \cong H^i_t(M) \). Then

\[
\text{Supp} \, H^i_{(x_1, \ldots, x_t)}(M) = \text{Supp} \, H^i_t(M) \subseteq \text{Supp} \, \frac{M}{IM}
\]

and for each \( 0 \leq i \leq t - 1 \), \( \dim \text{Supp} \, H^i_{(x_1, \ldots, x_t)}(M) \leq 1 \). By [1], clearly the \( R \)-module \( H^i_{(x_1, \ldots, x_{t-1})}(M) \) is \((x_1, \ldots, x_t)\)-cofinite. Since \( H^i_{(x_1, \ldots, x_t)}(M) = 0 \) for all \( i \geq t + 1 \), it follows from [15], that \( H^i_{(x_1, \ldots, x_t)}(M) \) is also \((x_1, \ldots, x_t)\)-cofinite. Consequently for each \( i \geq 0 \), the \( R \)-module \( H^i_{(x_1, \ldots, x_t)}(M) \) is \((x_1, \ldots, x_t)\)-Cofinite. Now, let \( x_{t+1} \in I \) be such that \( x_1, \ldots, x_{t+1} \) is \( I \)-filter regular sequence. Since \( x_{t+1} \in I \) and \( H^{t-1}_{(x_1, \ldots, x_{t})}(M) \cong H^{t-1}_I(M) \) is \( I \)-torsion, then \( H^t_{Rx_{t+1}}(H^{t-1}_{(x_1, \ldots, x_{t})}(M)) = 0 \). On the other hand by [17], the following exact sequence is hold: \( 0 \rightarrow H^t_{Rx_{t+1}}(H^{t-1}_{(x_1, \ldots, x_{t})}(M)) \rightarrow H^t_{(x_1, \ldots, x_{t+1})}(M) \rightarrow H^0_{Rx_{t+1}}(H^t_{(x_1, \ldots, x_{t})}(M)) \rightarrow 0 \). But, \( H^t_{(x_1, \ldots, x_{t})}(M) \cong H^t_I(M) \) and so by the above exact sequence, \( H^t_I(M) \cong H^0_{Rx_{t+1}}(H^t_{(x_1, \ldots, x_{t})}(M)) \). Since \( Rx_{t+1} \subseteq I \), it follows that

\[
H^t_I(H^t_{(x_1, \ldots, x_{t})}(M)) \subseteq H^0_{Rx_{t}}(H^t_{(x_1, \ldots, x_{t})}(M)).
\]

Also, \( H^0_{Rx_{t+1}}(H^t_{(x_1, \ldots, x_{t})}(M)) \cong H^t_I(M) \) is \( I \)-torsion and hence

\[
H^0_{Rx_{t+1}}(H^t_{(x_1, \ldots, x_{t})}(M)) \subseteq H^t_I(H^t_{(x_1, \ldots, x_{t})}(M)).
\]

Then

\[
H^t_I(M) \cong \Gamma_{Rx_{t+1}}(H^t_{(x_1, \ldots, x_{t})}(M)) = \Gamma_I(H^t_{(x_1, \ldots, x_{t})}(M)).
\]
Finally from the exact sequence
\[ 0 \to H^I_t(M) \cong H^0_I(H^I_{(x_1, \ldots, x_t)}(M)) \to H^I_{(x_1, \ldots, x_t)}(M) \]
and \((x_1, \ldots, x_t)\)-cofinitness of \(H^I_{(x_1, \ldots, x_t)}(M)\), we conclude that
\[ \text{Hom}_R \left( \frac{R}{(x_1, \ldots, x_t)}, H^I_1(M) \right) \]
is finitely generated. □

**Lemma 2.10.** Let \(M\) be an \(R\)-module and \(I\) be an ideal of \(R\) such that \(\text{Supp} M \subseteq V(I)\). Let \(x \in I\) be such that \(0 :_M x\) and \(M/xM\) are \(I\)-cominimax. Then so is \(M\).

*Proof.* The proof is similar to the proof of [15, Corollary 3.4]. □

**Theorem 2.11.** With the assumption of Theorem 2.9, the \(R\)-module \(H^I_1(M)\) is \((x_1, \ldots, x_t)\)-cominimax.

*Proof.* We prove by induction on \(t\). If \(t = 1\), then we set \(N = \frac{M}{\Gamma_I(M)}\) and so \(x_1\) is an \(N\)-regular element and \(H^I_1(N) \cong H^I_1(M)\).

Consider the exact sequence
\[ 0 \to N \xrightarrow{x_1} N \xrightarrow{\frac{x_1}{x_1N}} N \xrightarrow{\frac{x_1}{x_1N}} 0 \]
which implies that the following exact sequence
\[ \ldots \to H^I_1(\frac{N}{x_1N}) \to H^I_1(N) \xrightarrow{x_1} H^I_1(N) \to H^I_1(\frac{N}{x_1N}) \]
Clearly the \(R\)-module \(0 :_H^I_1(N) x_1\) is finitely generated, and \(Rx_1\)-cominimax. Set
\[ T = \{ P \in \text{Supp} H^I_1(N) \mid \dim \frac{R}{P} = 1 \}. \]
Then \((H^I_1(N))_P\) for all \(P \in T\) is Artinian and \(Rx_1\)-cofinite. Also \(T \subseteq \text{Ass} \frac{M}{IM}\) and so is finite. By argument in [1, Theorem 2.6], \(\frac{H^I_1(N)}{x_1H^I_1(N)}\) is minimax. Also \(\frac{H^I_1(N)}{x_1H^I_1(N)}\) and \(0 :_H^I_1(N) x_1\) are \(Rx_1\)-cominimax and hence \(H^I_1(N)\) is also \(Rx_1\)-cominimax.

Now, let \(t \geq 2\). Clearly \(x_1, \ldots, x_t\) is \(I\)-filter regular sequence over the \(R\)-module \(\frac{M}{\Gamma_I(M)}\). Now \(H^I_1(M) \cong H^I_1(\frac{M}{\Gamma_I(M)})\) and \(\frac{M}{\Gamma_I(M)}\) is a finitely generated \(I\)-torsion free \(R\)-module. We therefore assume in addition that \(\Gamma_I(M) = 0\). Since \(x_1 \notin \cup_{P \in \text{Ass} M \setminus V(I)} P = \cup_{P \in \text{Ass}(M)} P\), it follows that \((x_1, \ldots, x_t) \notin \cup_{P \in \text{Ass} M} P\).
Set $T := \{ P \in \text{Supp } H_{i-1}^i(M) \cup \text{Supp } H_i^i(M) \mid \dim \frac{R}{P} = 1 \}$. Hence $T \subseteq \text{Assh}_R \frac{M}{IM}$, and so $T$ is a finite set. Let $T = \{ P_1, \ldots, P_n \}$. Then for each $i \geq 0$, $\text{Supp } H_{i+1}^i(M_{P_k}) \subseteq \{ P_k R_{P_k} \}$, where $k = 1, 2, \ldots, n$. By [1], for each $t - 1 \leq k \leq t$, $H_{i+1}^i(M_{P_k})$ is $R_{P_k}$-Artinian and $(x_1, \ldots, x_t)R_{P_k}$-cofinite. Also

$$V((x_1, \ldots, x_t)R_{P_k}) \cap \text{Att}_{R_{P_k}} H_{i+1}^i(M_{P_k}) \subseteq V(P_k R_{P_k}).$$

Set

$$U := \cup_{t=1}^t \cup_{k=1}^n \{ q \in \text{Spec}(R) \mid q R_{P_k} \in \text{Att}_{R_{P_k}} (H_{i+1}^i(M_{P_k})) \}.$$

Therefore $U \cap V(x_1, \ldots, x_t) \subseteq T$. Since $(x_1, \ldots, x_t) \not\subseteq (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \text{Ass } M} P)$, it follows that there exists an element $z_1 \in (x_1, \ldots, x_t)$ such that $x_1 + z_1 \not\subseteq (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \text{Ass } M} P)$.

Assume that $y_1 = x_1 + z_1$, then $(x_1, \ldots, x_t) = (y_1, x_2, \ldots, x_t)$ and $y_1 \in I$ is an $I$-filter regular sequence.

Now if $(x_1, \ldots, x_t) = (y_1, x_2, \ldots, x_t) \subseteq \cup_{P \in (\text{Ass } R/\langle y_1 \rangle) \setminus V(I)} P$, then there exists $P \in (\text{Ass } R/\langle y_1 \rangle) \setminus V(I)$ such that $(x_1, \ldots, x_t) \subseteq P$.

Since $I \not\subseteq P$, it follows that $\frac{x_1}{1}, \ldots, \frac{x_t}{1} \in PR_P$ is a $R_P$-regular sequence and so \( \text{grade}(\frac{x_1}{1}, \ldots, \frac{x_t}{1}; R_P) = t \). On the other hand $PR_P \in \text{Ass } R/\langle y_1 \rangle$ and $(y_1, x_2, \ldots, x_t)R_P \subseteq PR_P$.

Then \( \text{grade}(\frac{y_1}{1}, x_2, \ldots, x_t)R_P, R_P) = 1 \) if $t \geq 2$, and so $(y_1, x_2, \ldots, x_t) \not\subseteq \cup_{P \in \text{Ass } R/\langle y_1 \rangle} P$. Hence there exists an element $z_2 \in (y_1, x_2, \ldots, x_t)$ such that $x_2 + z_2 \not\subseteq \cup_{P \in \text{Ass } R/\langle y_1 \rangle} P$. Again, we put $y_2 = x_2 + z_2$, then $(y_1, x_2, \ldots, x_t) = (y_1, y_2, x_3, \ldots, x_t)$. By the similer argument in the above, we see that there exist elements $y_1, \ldots, y_t \in I$ such that $(x_1, \ldots, x_t) = (y_1, \ldots, y_t)$ and $y_1, \ldots, y_t$ is an $I$-filter regular sequence for $M$.

The exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \xrightarrow{M} M \xrightarrow{y_1M} 0$$

induces a short exact sequence of local cohomology modules

$$0 \longrightarrow H_{i-1}^i(M) \underset{y_1H_{i-1}^i(M)}{\longrightarrow} H_{i-1}^i(M) \underset{y_1M \cdot y_1M}{\longrightarrow} 0 : H_{i}^i(M) y_1 \longrightarrow 0$$
By a similar proof in [1], we see that \( \frac{H_{H}^{t-1}(M)}{y_1 H_{H}^{t-1}(M)} \) is a minimax \( R \)-module.

Now, by induction hypothesis and since \( y_2, \ldots, y_t \) is an \( I \)-filter regular sequence for \( \frac{M}{y_1 M} \), we conclude that the \( R \)-module \( \frac{H_{H}^{t-1}(M)}{y_1 M} \) is \((y_2, \ldots, y_t)\)-cominimax. Also, we note that \((y_2, \ldots, y_t) \subseteq (y_1, \ldots, y_t)\) and also \( \text{Supp} \frac{H_{H}^{t-1}(M)}{y_1 M} \subseteq V(y_1, \ldots, y_t) \). Therefore \( \frac{H_{H}^{t-1}(M)}{y_1 M} \) is \((y_1, \ldots, y_t)\)-cominimax. Consequently by the above exact sequence \( 0 : H_{H}^{t}(M) \) \( y_1 \) is also \((y_1, \ldots, y_t)\)-cominimax. On the other hand by argument in [1, Theorem 2.6], the \( R \)-module \( \frac{H_{H}^{t}(M)}{y_1 H_{H}^{t}(M)} \) is minimax and hence is \((y_1, \ldots, y_t)\)-cominimax.

Finally, \( y_1 \in (y_1, \ldots, y_t) = (x_1, \ldots, x_t) \) and the \( R \)-modules \( 0 : H_{H}^{t}(M) \) \( y_1 \) and \( \frac{H_{H}^{t}(M)}{y_1 H_{H}^{t}(M)} \) are both \((x_1, \ldots, x_t)\)-cominimax. Thus by lemma 2.9, the \( R \)-module \( H_{H}^{t}(M) \) is also \((x_1, \ldots, x_t)\)-cominimax. \( \square \)

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REFERENCES


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FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

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