A GENERALIZATION OF PRIME HYPERIDEALS IN KRASNER HYPERRINGS

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ABSTRACT. In this paper, we give a characterization of new generalization of prime hyperideals in Krasner hyperrings by introducing 2-absorbing hyperideals. We study fundamental properties of 2-absorbing hyperideals on Krasner hyperrings and investigate some related results.

1. Introduction

Prime ideals play a significant role in commutative ring theory. Because of this importance, the concept of 2-absorbing ideals in a commutative ring was introduced by Badawi [2] as a generalization of prime ideals. After this, [8, 9, 10] have continued these studies and obtained several results. Recently, this notion is generalized to the hypercase by introducing the 2-absorbing hyperideals in a multiplicative hyperring [1]. In this paper, we introduce the notion of the 2-absorbing hyperideals on Krasner hyperrings and give some properties of such hyperideals.

Let us first recall some preliminary definitions.

Assume that $H$ is a non-empty set and $\mathcal{P}^*(H)$ is the set of all non-empty subsets of $H$. A hyperoperation on $H$ is a map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid. If $A$ and $B$ are non-empty subsets of $H$, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called

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Throughout this paper, by a hyperring we mean a Krasner hyperring. There are comprehensive references for hyperrings, for example see [3, 7]. In fact, different kinds of hyperrings are defined which one of them is Krasner hyperring described as follows [6]:

A Krasner hyperring is an algebraic structure $\langle R, +, \cdot \rangle$ satisfying the following axioms: (1) $(R, +)$ is a canonical hypergroup which means that (i) $(R, +)$ is a semihypergroup, i.e., $x + (y + z) = (x + y) + z$, for all $x, y, z \in R$, (ii) $x + y = y + x$, for all $x, y \in R$, (iii) There exists $0 \in R$ such that $0 + x = \{x\}$, for all $x \in R$, (iv) For all $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, (we write $-x$ for $x'$ and we call it the opposite of $x$), (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in R$; (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; (3) The multiplication is distributive with respect to the hyperoperation $+$. Throughout this paper, by a hyperring we mean a Krasner hyperring.

The meaning of center of a hyperring $(R, +, \cdot)$ is $Z(R) = \{x \in R \mid x \cdot y = y \cdot x, \text{ for all } y \in R\}$ and $R$ is called commutative if $Z(R) = R$ i.e., $(R, \cdot)$ is a commutative semigroup. A hyperring $(R, +, \cdot)$ is called hyperfield if $(R, \cdot)$ is a commutative monoid and all nonzero elements of $R$ are multiplicatively invertible. The identity element of the monoid $(R, \cdot)$ is called unit element of hyperring $(R, +, \cdot)$. For example, suppose that $\mathbb{K} := \{0, 1\}$ is a commutative monoid with the multiplication $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. The hyperaddition is given by $0 + 1 = 1 + 0 = 1$, $0 + 0 = 0$ and $1 + 1 = \{0, 1\}$. Then, $\mathbb{K}$ is a hyperfield called the Krasner hyperfield with unit element $1$ [5]. A hyperring $(R, +, \cdot)$ is called hyperdomain, if $R$ is a commutative hyperring with unit element and $xy = 0$ implies that $x = 0$ or $y = 0$, for all $x, y \in R$.

A non-empty subset $A$ of a hyperring $(R, +, \cdot)$ is called subhyperring of $R$ if $(A, +, \cdot)$ is itself a hyperring. A non-empty subset $I$ of a hyperring $R$ is called a hyperideal if and only if (1) $u, v \in I$ imply that $u - v \subseteq I$, for all $u, v \in I$, (2) $u \in I$ and $r \in R$ imply that $r \cdot u \in I$ and $u \cdot r \in I$. Remember here that $(I :_R x) = \{y \in R \mid y \cdot x \in I\}$, for all $x \in R$, is a hyperideal. A hyperideal $I$ is called prime if $xy \in I$ implies that $x \in I$ or $y \in I$. A prime hyperideal $P$ is said to be a minimal prime hyperideal over an ideal $I$ if it is minimal among all prime ideals containing $I$. Note that we do not exclude $I$ even if it is a prime ideal. A prime hyperideal is said to be a minimal prime hyperideal if it is a minimal prime ideal over the zero hyperideal. By applying the argument similar in spirit to the proof of Theorem 2.1 of [4], one can easily
show that if $I$ and $P$ are hyperideals of $R$ such that $I \subseteq P$ and $P$ is a minimal prime hyperideal of $I$, then, for all $x \in P$, there is $y \in R \setminus P$ and a nonnegative integer $n$ such that $yx^n \in I$.

A good homomorphism between two hyperrings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ is a map $f : R_1 \rightarrow R_2$ such that for all $x, y \in R_1$, we have $f(x +_1 y) = f(x) +_2 f(y)$, $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ and $f(0) = 0$. Let $f : R_1 \rightarrow R_2$ be a good homomorphism. The kernel of $f$ is defined as $\ker f = \{x \in R_1 \mid f(x) = 0\}$. It is trivial that $\ker f$ is a hyperideal of $R_1$. Note that a prime hyperideal of a commutative hyperring $R$ can be described as the kernel of a homomorphism from $R$ to the Krasner hyperfield $\mathbb{K}$ [5].

2. 2-absorbing hyperideals in Krasner hyperrings

In this section, we treat to the introducing 2-absorbing hyperideals on Krasner hyperrings and investigate more results with respect to such hyperideals.

**Definition 2.1.** A proper hyperideal $I$ of a hyperring $(R, +, \cdot)$ is called a 2-absorbing hyperideal if $a \cdot b \cdot c \in I$ implies that $a \cdot b \in I$ or $a \cdot c \in I$ or $b \cdot c \in I$, for all $a, b, c \in R$.

**Example 2.2.** Let $(G, \circ)$ be a group and $H = G \cup \{0, u, v\}$, where $0$ is an absorbing element under multiplication and $u, v$ are distinct orthogonal idempotents with

$$
\begin{align*}
    u \circ v &= v \circ u = 0; & u \circ u &= u; \\
    v \circ v &= v; & a \circ 0 &= 0 \circ a = 0, \text{ for all } a \in H; \\
    u \circ g &= g \circ u = u; & v \circ g &= g \circ v = v, \text{ for all } g \in G.
\end{align*}
$$

If we define hyperoperation $\oplus$ on $H$ as follows:

$$
\begin{align*}
    a \oplus 0 &= 0 \oplus a = \{a\}; & a \oplus a &= \{0, a\}, \text{ for all } a \in H; \\
    a \oplus b &= b \oplus a = H \setminus \{0, a, b\}, \text{ for all } a, b \in H \setminus \{0\} \text{ and } a \neq b.
\end{align*}
$$

Then, $(H, \oplus, \circ)$ is a Krasner hyperring [3]. Put $I = \{0, u\}$ and $J = \{0\}$. Obviously, $I$ and $J$ are 2-absorbing hyperideals. The hyperideal $I$ is prime but $J$ is not prime, because $u \circ v = 0 \in J$ while $u, v \notin J$.

**Example 2.3.** Let $(R, +, \cdot)$ be a hyperdomain and

$$
M = \left\{ \begin{pmatrix} x_1 & x_2 \\
0 & 0 \end{pmatrix} \mid x_1, x_2 \in R \right\}.
$$
Put $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in R \right\}$ and define the hyperoperation $\oplus$ and the operation $\odot$ on $M$ as

$$\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} x'_1 & x'_2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix} \mid y_i \in x_i + x'_i, 1 \leq i \leq 2 \right\}$$

and

$$\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} x'_1 & x'_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_1 \cdot x'_1 & x_1 \cdot x'_2 \\ 0 & 0 \end{pmatrix}.$$

Then, $(M, \oplus, \odot)$ is a Krasner hyperring and $I$ is a 2-absorbing hyperideal of $M$.

Note that by the same argument of Theorem 2.8 of [1], one can show that a nonzero proper hyperideal $I$ of a hyperring $R$ is a 2-absorbing hyperideal if and only if whenever $I_1 \cdot I_2 \cdot I_3 \subseteq I$, for some hyperideals $I_1, I_2, I_3$ of $R$, then $I_1 \cdot I_2 \subseteq I$ or $I_2 \cdot I_3 \subseteq I$ or $I_1 \cdot I_3 \subseteq I$.

From now on, the hyperring $(R, +, \cdot)$ is commutative with unit element. Also, we may use $xy$ instead of $x \cdot y$.

**Theorem 2.4.** Let $I$ be a 2-absorbing hyperideal of $R$. Then, one of the following statements is valid:

1. $\sqrt{I} = P$ is a prime hyperideal of $R$ and $P^2 \subseteq I$;
2. $\sqrt{I} = P_1 \cap P_2$, $P_1 P_2 \subseteq I$ and $(\sqrt{I})^2 \subseteq I$, where $P_1$ and $P_2$ are the only distinct prime hyperideals of $R$ that are minimal over $I$.

**Proof.** We prove this statement in three steps:

Step 1: $\sqrt{I}$ is a 2-absorbing hyperideal of $R$.

Suppose that $x, y, z \in R$ such that $xyz \in \sqrt{I}$. By assumption, $(xyz)^2 \in I$. Thus, $x^2 y^2 z^2 \in I$ and this implies that $(xy)^2 = x^2 y^2 \in I$ or $(xz)^2 = x^2 z^2 \in I$ or $(yz)^2 = y^2 z^2 \in I$. Therefore, at least one of $xy, xz$ and $yz$ belongs to $\sqrt{I}$.

Step 2: There are at most two distinct prime hyperideals of $R$ that are minimal over $I$.

Suppose that $P_1$ and $P_2$ are distinct prime hyperideals of $R$ that are minimal over $I$. Then, there are $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. Also, there exist $c_2 \in R \setminus P_1$, $c_1 \in R \setminus P_2$ and $m, n \in \mathbb{N}$ such that $c_2 x_1^m, c_1 x_2^n \in I$. This implies that $c_2 x_1, c_1 x_2 \in I \subseteq P_1 \cap P_2$, because $I$ is a 2-absorbing hyperideal. Consequently, $c_1 \in P_1 \setminus P_2$ and $c_2 \in P_2 \setminus P_1$. Hence, $(c_1 + c_2) \cap P_1 = \emptyset$, since if $t \in (c_1 + c_2) \cap P_1$, then $c_2 \in -c_1 + t \subseteq P_1$ which contradicts $c_2 \not\in P_1$. In the same way, $(c_1 + c_2) \cap P_2 = \emptyset$. Therefore, for all $t \in c_1 + c_2$ we have $tx_2 \not\in P_1$ and $tx_1 \not\in P_2$ which lead
to $tx_1, tx_2 \not\in I$. On the other hand, $(c_1 + c_2)x_1x_2 \subseteq I$. Thus, for all $t \in c_1 + c_2$ we get $tx_1x_2 \in I$ and this implies that $x_1x_2 \in I$.

Now, suppose that $P_3$ is a prime hyperideal of $R$ that is minimal over $I$ and $P_3 \neq P_1, P_2$. Consequently, there are $y_1 \in P_1 \setminus (P_2 \cup P_3)$ and $y_2 \in P_2 \setminus (P_1 \cup P_3)$. Then, by the previous argument $y_1y_2 \in I \subseteq P_3$ which leads to $y_1 \in P_3$ or $y_2 \in P_3$, a contradiction.

Step 3: In this step, we prove the principle assertion of Theorem. Suppose that $x, y \in \sqrt{I}$. Then, $x^2, y^2 \in I$ and so $x(x + y)y \subseteq I$. Therefore, for all $t \in x + y$, we have $xt \in I$ or $xy = t \in I$, because $I$ is a 2-absorbing hyperideal. If $xt \in I$, then $xt \in x(x + y) = x^2 + xy$ and consequently $xy = x^2 + xt \subseteq I$. Similarly, $ty \in I$ yields $xy \in I$. Therefore, we have $(\sqrt{I})^2 \subseteq I$. By Steps (1) and (2), $\sqrt{I} = P$ is a prime hyperideal of $R$ or $\sqrt{I} = P_1 \cap P_2$, where $P_1$ and $P_2$ are the only distinct prime hyperideals of $R$ that are minimal over $I$. If $\sqrt{I} = P_1 \cap P_2$ then $P_2 = (\sqrt{I})^2 \subseteq I$. If $\sqrt{I} = P_1 \cap P_2$, then for all $y \in \sqrt{I}$, $z_1 \in P_1 \setminus P_2$ and $z_2 \in P_2 \setminus P_1$ we have $y + z_1 \subseteq P_1 \setminus P_2$. By the same argument of Step 2, we get $z_1z_2 \in I$ and $(y + z_1)z_2 \subseteq I$. Thus, for all $s \in yz_2 + z_1z_2$, we have $yz_2 \in s - z_1z_2 \subseteq I$. Similarly, $yz_1 \in I$ and this implies that $P_1P_2 \subseteq I$.

**Theorem 2.5.** Let $I$ be a hyperideal of $R$. Then, $I$ is a 2-absorbing hyperideal of $R$ if and only if $(I : R x)$ is a prime hyperideal of $R$ containing $\sqrt{I}$, for all $x \in \sqrt{I} \setminus I$.

**Proof.** By Theorem 2.4, either $\sqrt{I} = P$ or $\sqrt{I} = P_1 \cap P_2$, where $P$ is a prime hyperideal and $P_1, P_2$ are nonzero distinct prime hyperideals of $R$ that are minimal over $I$. We prove the statement for the case $\sqrt{I} = P_1 \cap P_2$ and a similar argument implies the assertion for the case $\sqrt{I} = P$.

Suppose that $I$ is a 2-absorbing hyperideal of $R$. According to Theorem 2.4, we conclude $xP_1, xP_2 \subseteq I$, for all $x \in \sqrt{I} \setminus I$. This means that $P_1, P_2 \subseteq (I : R x)$ and consequently $\sqrt{I} \subseteq (I : R x)$. Assume that $yz \in (I : R x)$, where $y, z \in R$ and $x \in \sqrt{I} \setminus I$. Clearly, the statement is valid when $y \in P_1 \cup P_2$ or $z \in P_1 \cup P_2$. Then, we prove it for $y, z \not\in P_1 \cup P_2$. In this case, we have $yz \not\in P_1 \cap P_2 = \sqrt{I}$ that leads to $yz \not\in I$. Hence, by assumption we get $y \in (I : R x)$ or $z \in (I : R x)$ which implies that $(I : R x)$ is a prime hyperideal.

Now, suppose that $(I : R x)$ is a prime hyperideal, for all $x \in \sqrt{I} \setminus I$. In order to prove that $I$ is a 2-absorbing hyperideal, assume that $xyz \in I$, where $x, y, z \in R$. Then, $yz \in (I : R x)$. Obviously, at least one of $x, y, z$ belongs to $(P_1 \cup P_2) \setminus I$. For proving the assertion, without loss of generality suppose that $x \in (P_1 \cup P_2) \setminus I$. In this case either
$x \in \sqrt{I} \setminus I$ or $x \in (P_1 \cup P_2) \setminus \sqrt{I}$. If $x \in \sqrt{I} \setminus I$, then by the hypothesis we get $y \in (I : R x)$ or $z \in (I : R x)$. Consequently, $yx \in I$ or $zx \in I$ which implies that $I$ is a 2-absorbing hyperideal. If $x \in (P_1 \cup P_2) \setminus \sqrt{I}$, then $(I : R x_1) = P_2$ and $(I : R x_2) = P_1$, for all $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. Similar to the previous argument, we find that $I$ is a 2-absorbing hyperideal.

\[\square\]

**Theorem 2.6.** Let $I$ be a 2-absorbing hyperideal and $P = P_1$ and $P_2$ be prime hyperideals of $R$. Then,

1. If $\sqrt{I} = P$, then $(I : R x)$ is a 2-absorbing hyperideal of $R$, for all $x \in R \setminus P$ with $\sqrt{(I : R x)} = P$ and $\Omega = \{(I : R x) \mid x \in R\}$ is a totally ordered set;
2. If $\sqrt{I} = P_1 \cap P_2$, then $(I : R x)$ is a 2-absorbing hyperideal of $R$, for all $x \in R \setminus (P_1 \cup P_2)$ with $\sqrt{(I : R x)} = P_1 \cap P_2$ and $\Omega = \{(I : R x) \mid x \in R \setminus (P_1 \setminus P_2)\}$ is a totally ordered set;
3. If $\sqrt{I} = P_1 \cap P_2$, then $(I : R x) = P_2$, for all $x \in P_1 \setminus P_2$ and $(I : R x) = P_1$, for all $x \in P_2 \setminus P_1$.

**Proof.** The proof is similar to Theorem 2.5 of [1]. \[\square\]

**Theorem 2.7.** Let $I$ be a 2-absorbing hyperideal of $R$ such that $I \neq \sqrt{I}$. Then,

1. If $x \in \sqrt{I} \setminus I$ and $y \in R$ such that $yx \notin I$, then $(I : R yx) = (I : R x)$;
2. If $x, y \in \sqrt{I} \setminus I$, then $(I : R fx + dy) = (I : R x)$, for all $f, d \in R$ such that $fd \notin (I : R x)$. In particular, $(I : R x + y) = (I : R x)$.

**Proof.** (1) Suppose that $c \in (I : R yx)$, where $x \in \sqrt{I} \setminus I$ and $y \in R$. Then, $cy \in (I : R x)$ which means that $c \in (I : R x)$, by Theorem 2.5. Therefore, $(I : R yx) \subseteq (I : R x)$. It is clear that $(I : R x) \subseteq (I : R yx)$ and consequently the statement is valid.

(2) Suppose that $x, y \in \sqrt{I} \setminus I$. Then, $(I : R x) \subseteq (I : R y)$ or $(I : R y) \subseteq (I : R x)$, by Theorem 2.6. In order to establish the assertion, without loss the generality, assume that $(I : R x) \subseteq (I : R y)$ which leads to $(I : R x) \subseteq (I : R y) \subseteq (I : R dy)$ and $(I : R x) \subseteq (I : R fx)$. Therefore, for all $t \in (I : R x)$ we get $t(dy + fx) \subseteq I$ and so $(I : R x) \subseteq (I : R dy + fx)$. For proving equality, suppose that there exists $s \in dy + fx$ such that $(I : R x) \neq (I : R s)$. By applying Theorem 2.6, there exists $z \in (I : R y) \cap (I : R s)$ such that $z \notin (I : R x)$, because $(I : R x) \subseteq (I : R y)$ and $(I : R x) \subseteq (I : R dy + fx)$. Since $zs \in z(dy + fx)$, hence $zf \in z(dy + zs) \subseteq I$ which means that $zf \in (I : R x)$. Therefore, $z \in (I : R x)$ or $f \in (I : R x)$ and this is a contradiction. \[\square\]
Definition 2.8. Let $I$ be a nonzero proper hyperideal of $R$ and
\[ Z_R(R/I) = \{ r + I \in R/I \mid \exists s \in R \setminus I \text{ such that } rs \in I \}. \]
Then, $I$ is called Primal if $Z_R(R/I)$ is a prime hyperideal of $R$ containing $I$.

Theorem 2.9. Let $I$ be a 2-absorbing hyperideal of $R$ such that $I \neq \sqrt{I}$. Then, $I$ is a Primal hyperideal of $R$.

Proof. First, we show that $Z_R(R/I) = Q/I$, where $Q = \bigcup_{x \in (\sqrt{I} \setminus I)} (I :_R x)$. For this purpose, suppose that $a + I \in Q/I$. Then, there exists $x \in \sqrt{I} \setminus I$ such that $a \in (I :_R x)$. Therefore, $ax \in I$ which follows that $a + I \in Z_R(R/I)$. For proving $Z_R(R/I) \subseteq Q/I$, assume that $a + I \in Z_R(R/I)$, where $a \notin I$. Then, there is $b \in R \setminus I$ such that $ab \in I$. By Theorem 2.4, we can distinguish two cases:

Case 1: $\sqrt{I} = P$ is a hyperideal of $R$. Then, we have $ab \in P$ and consequently $a \in P \setminus I$ or $b \in P \setminus I$. Therefore, $a \in (I :_R a)$ or $a \in (I :_R b)$ which implies that $a + I \in Q/I$.

Case 2: $\sqrt{I} = P_1 \cap P_2$, where $P_1$ and $P_2$ are the only distinct prime hyperideals of $R$ that are minimal over $I$. If $a \in \sqrt{I} \setminus I$ or $b \in \sqrt{I} \setminus I$, then by applying the same argument as for Case (1), we find $a + I \in Q/I$. Now, suppose that $a, b \notin \sqrt{I} \setminus I$. Therefore, $a$ belongs to $P_1 \setminus P_2$ or $P_2 \setminus P_1$ and consequently $a \in (I :_R b)$, by Theorem 2.5. Hence, $a + I \in Q/I$ which leads to $Z_R(R/I) \subseteq Q/I$.

Thus in both cases, we have $Z_R(R/I) = Q/I$ as desired. Moreover, since $I \neq \sqrt{I}$, then Theorem 2.6 implies that $\Omega = \{(I :_R x) \mid x \in \sqrt{I} \setminus I\}$ is a set of linear ordered (prime) hyperideals of $R$. Therefore, $Z_R(R/I) = \bigcup_{(I :_R x) \in \Omega} ((I :_R x)/I)$ is a hyperideal of $R/I$. \hfill \Box

Theorem 2.10. Let $R'$ be a commutative hyperring with unit element and $\varphi : R \longrightarrow R'$ be a good homomorphism.

1. If $I'$ is a 2-absorbing hyperideal of $R'$, then $\varphi^{-1}(I')$ is a 2-absorbing hyperideal of $R$;
2. If $\varphi$ is an epimorphism and $I$ is a 2-absorbing hyperideal of $R$ containing $\ker \varphi$, then $\varphi(I)$ is a 2-absorbing hyperideal of $R'$.

Proof. (1) Suppose that $abc \in \varphi^{-1}(I')$, then $\varphi(a)\varphi(b)\varphi(c) \in I'$. Therefore, at least one of the $\varphi(ab)$, $\varphi(bc)$ and $\varphi(ac)$ belongs to $I'$ which implies that $ab \in \varphi^{-1}(I')$ or $bc \in \varphi^{-1}(I')$ or $ac \in \varphi^{-1}(I')$.

(2) Assume that $a', b', c' \in R'$ such that $a'b'c' \in \varphi(I)$. Then, there are $a, b, c \in R$ such that $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(c) = c'$. Therefore, $\varphi(abc) = a'b'c' \in \varphi(I)$ which deduce that there is $i \in I$ such that
Let \( I \) and \( J \) be distinct proper hyperideals of \( R \). If \( J \subseteq I \) and \( I \) is a 2-absorbing hyperideal of \( R \), then \( I/J \) is a 2-absorbing hyperideal of \( R/J \).

**Theorem 2.12.** Let \( R_1, R_2 \) be Krasner hyperrings and \( R = R_1 \times R_2 \).

1. If \( I_1 \) (\( I_2 \), respectively) is a 2-absorbing hyperideal of \( R_1 \) (\( R_2 \), respectively), then \( I_1 \times R_2 \) (\( R_1 \times I_2 \), respectively) is a 2-absorbing hyperideal of \( R \);
2. If \( J \) is a 2-absorbing hyperideal of \( R \), then either \( J = I_1 \times R_2 \) (\( J = R_1 \times I_2 \), respectively), where \( I_1 \) (\( I_2 \), respectively) is a 2-absorbing hyperideal of \( R_1 \) (\( R_2 \), respectively) or \( I = I_1 \times I_2 \), where \( I_1 \) (\( I_2 \), respectively) is a prime hyperideal of \( R_1 \) (\( R_2 \), respectively).

**Proof.** (1) It is straightforward.

(2) Suppose that \( J \) is a proper 2-absorbing hyperideal of \( R \). Then, \( J = I_1 \times I_2 \), where for \( i = 1, 2 \) we have \( I_i \) is a hyperideal of \( R_i \). Assume that \( I_2 = R_2 \) and \( R' = R/(\{0\} \times R_2) \). Therefore, \( I_1 \neq R_1 \) and \( J' = J/(\{0\} \times R_2) \) is a 2-absorbing hyperideal of \( R' \), by Corollary 2.11. It follows that \( I_1 \) is a 2-absorbing hyperideal of \( R_1 \), since \( R' \cong R_1 \) and \( I_1 \cong J' \). In the same way, \( I_1 = R_1 \) implies that \( I_2 \) is a 2-absorbing hyperideal of \( R_2 \).

For completing the proof it is enough to show that if \( I_1 \neq R_1 \) and \( I_2 \neq R_2 \), then \( I_i \) is a prime hyperideal of \( R_i \), where \( i = 1, 2 \). Assume that at least one of \( I_i \) is not prime, e.g. \( I_1 \). Therefore, there are \( a, b \in R_1 \) such that \( ab \in I_1 \) but \( a, b \not\in I_1 \). Putting \( x = (a, 1), y = (1, 0) \) and \( z = (b, 1) \), we give \( xyz = (ab, 0) \in J \) while \( xy = (a, 0), xz = (ab, 1), yz = (b, 0) \) do not belong to \( J \) and this is a contradiction to the assumption. \( \square \)

**Theorem 2.13.** Let \( I \) be a hyperideal of \( R \) and \( S \) be a multiplicatively closed subset of \( R \). In addition, let \( S^{-1}R \) be the hyperring of quotients of \( R \).

1. If \( I \) is a 2-absorbing hyperideal of \( R \) and \( S \cap I = \emptyset \), then \( S^{-1}I \) is a 2-absorbing hyperideal of \( S^{-1}R \);
2. If \( S^{-1}I \) is a 2-absorbing hyperideal of \( S^{-1}R \) and \( S \cap Z_R(R/I) = \emptyset \), then \( I \) is a 2-absorbing hyperideal of \( R \).
A GENERALIZATION OF PRIME HYPERIDEALS

Proof. (1) Suppose that \( a, b, c \in R \) and \( s, t, k \in S \) such that \((a/s)(b/t)(c/k) \in S^{-1}I\). Then, there exists \( u \in S \) such that \( uabc \in I \). Hence, \( uab \in I \) or \( uac \in I \) or \( bc \in I \), by hypothesis. If \( uab \in I \), then \((a/s)(b/t) = (uab)/(ust) \in S^{-1}I\). Also, \( uac \in I \) implies that \((a/s)(c/k) = (uac)/(usk) \in S^{-1}I\) and \( bc \in I \). Therefore, \((b/t)(c/k) \in S^{-1}I\). By the above result, \( S^{-1}I \) is a 2-absorbing hyperideal.

(2) Suppose that \( a, b, c \in I \) such that \( abc \in I \). In this case, we have \((abc)/1 = (a/1)(b/1)(c/1) \in S^{-1}I\). Hence, \((a/1)(b/1) \in S^{-1}I \) or \((b/1)(c/1) \in S^{-1}I \) or \((a/1)(c/1) \in S^{-1}I \), since \( S^{-1}I \) is a 2-absorbing hyperideal. If \((a/1)(b/1) \in S^{-1}I \), then there exists \( u \in S \) such that \( uab \in I \). This implies that \( ab \in I \), since \( S \cap Z_R(R/I) = \emptyset \).

Similarly, \((b/1)(c/1) \in S^{-1}I \) ((\(a/1)(c/1) \in S^{-1}I \), respectively) which leads to \( bc \in I \) (\(ac \in I \), respectively). Consequently, \( I \) a 2-absorbing hyperideal.

\[\text{Definition 2.14.} \quad \text{A proper hyperideal } I \text{ of } R \text{ is called irreducible precisely if } I \text{ can not be expressed as the intersection of two strictly larger hyperideals of } R.\]

The following theorem shows the relationship between irreducible and 2-absorbing hyperideals.

\[\text{Theorem 2.15.} \quad \text{Let } I \text{ be an irreducible hyperideal of } R \text{ and } P = P_1, P_2 \text{ be distinct prime hyperideals of } R.\]

\begin{enumerate}
\item If \( \sqrt{I} = P \), then \( I \) is a 2-absorbing hyperideal if and only if \( P^2 \subseteq I \) and \( (I :_Rx) = (I :Rx^2), \) for all \( x \in R \setminus P \);
\item If \( \sqrt{I} = P_1 \cap P_2 \), then \( I \) is a 2-absorbing hyperideal if and only if \( P_1P_2 \subseteq I \) and \( (I :_Rx) = (I :Rx^2) \), for all \( x \in R \setminus P_1 \cap P_2 \).
\end{enumerate}

Proof. (1) For proving the necessity part, it is only necessary to check \( (I :Rx^2) \subseteq (I :Rx) \), for all \( x \in R \setminus P \). Because, it is clear \( (I :Rx) \subseteq (I :Rx^2) \) and \( P^2 \subseteq I \), by Theorem 2.4.

Suppose that \( y \in (I :Rx^2) \). Then, \( yx \in I \) or \( x^2 \in I \). If \( x^2 \in I \), then \( x \in P \) and this is a contradiction. Then, \( yx \in I \) which implies that \( y \in (I :Rx) \) and consequently \( (I :Rx^2) \subseteq (I :Rx) \) as desired.

For establishing the sufficiency part, assume that \( x, y, z \in R \) such that \( xyz \in I \) and \( xy \notin I \). We show that either \( xz \in I \) or \( yz \in I \). From \( xy \notin I \), it follows that \( x \notin P \) or \( y \notin P \) and so \( (I :Rx) = (I :Rx^2) \) or \( (I :Ry) = (I :Ry^2) \), respectively. Without loss of generality, suppose that \( (I :Rx) = (I :Rx^2) \). For completing the proof as a contradiction, assume that \( xz \notin I \) and \( yz \notin I \). Consider \( a \in (I + xz) \cap (I + yz) \) which follows that there are \( a_1, a_2 \in I \) and \( r_1, r_2 \in R \) such that \( a \in (a_1 + r_1xz) \cap (a_2 + r_2yz) \). Consequently, \( ax \in a_1x + r_1x^2z \) and \( ax \in a_2x + r_2yzx \subseteq I \) which lead to \( r_1x^2z \in -a_1x + ax \subseteq I \).
Therefore, \( r_1z \in (I :_R x^2) = (I :_R x) \), by assumption. This implies that \( a \in a_1 + r_1xz \subseteq I \). Then, \( < I + xz > \cap < I + yz > \subseteq I \) and so \( < I + xz > \cap < I + yz > = I \) which contradicts irreducibility of \( I \).

(2) The proof is similar to Part (1). \( \square \)

In the process of proving the next theorem, we need the following lemma.

**Lemma 2.16.** Let \( P_1, P_2, \ldots, P_n \), where \( n \geq 2 \), be hyperideals of \( R \) such that at most two of them are not prime. Furthermore, let \( S \) be an additive canonical subhypergroup of \( R \) which is closed under multiplication and \( S \subseteq \bigcup_{i=1}^{n} P_i \). Then, there exists \( 1 \leq j \leq n \) such that \( S \subseteq P_j \).

**Proof.** We prove this statement by induction on \( n \). First, consider for \( n = 2 \) that is \( S \subseteq P_1 \cup P_2 \). As a contradiction, assume that \( S \not\subseteq P_1 \) and \( S \not\subseteq P_2 \). Then, there exists \( a_j \in S \setminus P_j \), where \( j = 1, 2 \). Therefore, the hypothesis leads to \( a_1 \in P_2 \) and \( a_2 \in P_1 \). On the other hand, \( a_1 + a_2 \not\subseteq S \subseteq P_1 \cup P_2 \) and so for all \( t \in a_1 + a_2 \) we have \( t \) belongs to either \( P_1 \) or \( P_2 \). Since \( a_1 \in \{a_1\} = a_1 + 0 \subseteq (a_1 + a_2) - a_2 \), then there exists \( t \in a_1 + a_2 \) such that \( a_1 \in t - a_2 \). By the above results, if \( t \in P_1 \), then \( a_1 \in P_1 \). Also, if \( t \in P_2 \), then \( a_2 \in -t - a_1 \subseteq P_2 \), which is a contradiction in two cases. Thus we must have \( S \subseteq P_1 \) or \( S \subseteq P_2 \). Now, suppose that \( k \geq 2 \) and our assertion is valid for \( n = k \). For completing the proof, assume that \( n = k + 1 \), where \( k \geq 2 \). Thus, we have \( S \subseteq \bigcup_{i=1}^{k+1} P_i \) and since at most two of the \( P_i \) are not prime, we can assume that they have been indexed in such a way that \( P_{k+1} \) is prime.

We claim that there is \( 1 \leq j \leq k \) such that \( S \subseteq \bigcup_{i=1}^{k+1} P_i \setminus_{i \neq j} P_i \). For proving this claim as a contradiction suppose that \( S \not\subseteq \bigcup_{i=1}^{k+1} P_i \setminus_{i \neq j} P_i \), for all \( 1 \leq j \leq k \). It follows that for all \( 1 \leq j \leq k \), there exists \( a_j \in S \setminus \bigcup_{i=1}^{k+1} P_i \) which implies that \( a_j \in P_j \), by hypothesis. Moreover, since \( P_{k+1} \in \text{Spec}(R) \), we conclude that \( a_1 \cdots a_k \not\in P_{k+1} \). Consequently, \( a_1 \cdots a_k \in \bigcap_{i=1}^{k} P_i \setminus P_{k+1} \) and \( a_{k+1} \in P_{k+1} \setminus \bigcup_{i=1}^{k} P_i \). Now consider the element \( b \in a_1 \cdots a_k + a_{k+1} \).

If \( b \in P_{k+1} \), then \( a_1 \cdots a_k \subseteq b - a_{k+1} \subseteq P_{k+1} \) and this is a contradiction. Therefore, \( b \) does not belong to \( P_{k+1} \). Also, we can not have \( b \in P_j \),
where \(1 \leq j \leq k\), for that would imply \(a_{k+1} \in b - a_1 \cdots a_k \subseteq P_j\), again a contradiction. But \(b \in S\), since for all \(1 \leq j \leq k\) we have \(a_j \in S\), which leads to a contradiction to the hypothesis that \(S \subseteq \bigcup_{i=1}^{k+1} P_i\). It follows that the statement is valid. In fact, there is \(1 \leq j \leq k+1\) such that \(S \subseteq \bigcup_{i \neq j}^{k+1} P_i\). By applying the inductive hypothesis, we deduce that \(S \subseteq P_i\), where \(1 \leq i \leq k + 1\).

\[ \square \]

**Theorem 2.17.** Let \(I_1, I_2, \ldots, I_n\) be \(2\)-absorbing hyperideals of \(R\) and \(I\) be a hyperideal of \(R\) such that \(I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n\). Then, there exists \(1 \leq i \leq n\) such that \(I^2 \subseteq I_i\).

**Proof.** First, we show that there exists \(1 \leq i \leq n\) such that \(\sqrt{I} \subseteq \sqrt{I_i}\). By Theorem 2.4, we can assume that they have been indexed in such a way that \(\sqrt{I_i} = p_i\) and \(\sqrt{I_j} = p_{j,1} \cap p_{j,2}\), for all \(1 \leq i \leq k\) and \(k + 1 \leq j \leq n\), where \(p_i, p_{j,1}, p_{j,2}\) are prime hyperideals of \(R\). Then, \(\sqrt{I} \subseteq p_1 \cup p_2 \cup \cdots \cup p_k \cup (p_{k+1,1} \cap p_{k+1,2}) \cup \cdots \cup (p_{n,1} \cap p_{n,2})\) which follows that \(\sqrt{I} \subseteq p_1 \cup p_2 \cup \cdots \cup p_k \cup p_{k+1,t_{k+1}} \cup \cdots \cup p_{n,t_n}\), where \(t_{k+1}, \ldots, t_n \in \{1,2\}\). Therefore by applying Lemma 2.16, we find that \(\sqrt{I} \subseteq p_i\) or \(\sqrt{I} \subseteq p_{j,t_s}\), for some \(1 \leq i \leq k\), \(k + 1 \leq j \leq n\) and \(t_s \in \{1,2\}\). If \(\sqrt{I} \subseteq p_{j,t_s}\), where \(k + 1 \leq j \leq n\), \(t_s \in \{1,2\}\), then \(\sqrt{I} \subseteq p_{j,t_s} \subseteq \bigcup_{j=k+1}^{n} p_{j,1}\). We may assume that \(\sqrt{I} \subseteq \bigcap_{j=k+1}^{s} p_{j,1}\) and \(\sqrt{I} \subseteq \bigcup_{j=s+1}^{n} p_{j,1}\), where \(k + 1 \leq s \leq n\). On the other hand, \(\sqrt{I} \subseteq p_{k+1,1} \cup \cdots \cup p_{s,2} \cup p_{s+1,1} \cup \cdots \cup p_{n,1}\). Therefore, \(\sqrt{I} \subseteq p_{j,2}\), for some \(k + 1 \leq j \leq s\), by Lemma 2.16. Hence, \(\sqrt{I} \subseteq p_{j,1} \cap p_{j,2} = \sqrt{I_j}\), where \(k + 1 \leq j \leq s\). Then, in general there is \(1 \leq i \leq n\) such that \(\sqrt{I} \subseteq \sqrt{I_i}\) which leads to \(I^2 \subseteq (\sqrt{I})^2 \subseteq (\sqrt{I_i})^2\). By applying Theorem 2.4, we get \(I^2 \subseteq I_i\). \[ \square \]

**References**


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A GENERALIZATION OF PRIME HYPERIDEALS IN KRASNER HYPERRINGS

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