A CHARACTERIZATION FOR METRIC TWO-DIMENSIONAL GRAPHS AND THEIR ENUMERATION

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ABSTRACT. The metric dimension of a connected graph $G$ is the minimum number of vertices in a subset $B$ of $G$ such that all other vertices are uniquely determined by their distances to the vertices in $B$. In this case, $B$ is called a metric basis for $G$. The basic distance of a metric two-dimensional graph $G$ is the distance between the elements of $B$. Giving a characterization for those graphs whose metric dimensions are two, we enumerate the number of $n$ vertex metric two-dimensional graphs with the basic distance 1.

1. INTRODUCTION

Let $G = (V, E)$ be a connected simple graph. For two vertices $u$ and $v$ of $G$, the distance $d_G(x, y)$ or $d(x, y)$ of $x$ and $y$ is the length of a minimum path connecting $x$ to $y$. For a subset $R = \{r_1, \ldots, r_k\}$ of $V$ and a vertex $v$, the representation of $v$ with respect to $R$ is the $k$-tuple $\langle v|R \rangle = (d(v, r_1), \ldots, d(v, r_k))$. The subset $R$ is called a resolving set for $G$ if any vertex has a unique representation with respect to $R$. A resolving set $B$ of $V$ is called a metric basis for $G$ if it has the minimum possible number of elements for a resolving set. The metric dimension $G$, denoted by $\dim_M(G)$, is then equal to this minimum number. For a study about these notions, we refer the reader to [4] and [8].

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As a simple known fact, $\dim_M(G) = 1$ if and only if $G$ is a path. The metric dimension of an $n$ vertex graph $G$ is $n - 1$ if and only if $G$ is the complete graph $K_n$; see [3].

The concept of a resolving set has various applications in different areas including network discovery and verification [1], problems of pattern recognition and image processing [6], robot navigation [5], mastermind game [2], and combinatorial search and optimization [7].

2. A Characterization for $\dim_M(G) = 2$

In this section, we aim to characterize all two metric dimensional graphs, but prior to this we need to extend the notion of a path.

**Definition 2.1.** Let $x$ and $d$ be two positive integers with $x \geq d$ and let $y$ be a nonnegative integer. An extended path $\mathcal{P}(x,y,d)$ of the length $x$, width $y$, and height $2d + 1$ is a simple graph with the following properties:

i. $V(\mathcal{P}) = \bigcup_{i=0}^{x} P_i$, where $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \ldots, v_{i,i+d}\}$ for $0 \leq i \leq y$ and $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \ldots, v_{i,i+d-1}\}$ for $y + 1 \leq i \leq x$;

ii. neighbors of $v_{i,j}$ are $v_{k,\ell}$ with $|i - k| \leq 1$ and $|j - \ell| \leq 1$.

**Example 2.2.** As an example, the generalized path $\mathcal{P}(7, 4, 1)$ has vertices of the form

```
   77
   66   76
   45   55   65
   34   44   54
   23   33   43
   12   22   32
   01   11   21
   10
```

and there is an edge between any two vertices, which are horizontally, vertically, or diagonally adjacent. Whence any horizontal, vertical, or diagonal line is a path. Here, $P_i$’s are vertical lines numbered from left to right by $P_0, P_1, \ldots, P_7$. The length of the first diagonal path from top is $y = 4$, the left coordinate of any vertex in the last path is $x = 7$, and the left coordinate of the only vertex in the first horizontal path from down is $d = 1$.

As another example, the generalized path $\mathcal{P}(6, 3, 4)$ has vertices of the form
and there is an edge between any two vertices which are horizontally, vertically, or diagonally adjacent.

**Definition 2.3.** Let $G$ be a metric two-dimensional graph with the metric basis $B = \{a, b\}$. Then $d(a, b)$ is called the basic distance of $G$ with respect to $B$ and is denoted by $BD_B(G)$.

**Proposition 2.4.** Let $x$ and $d$ be two positive integers with $x \geq d$ and let $y$ be a nonnegative integer. If $(x, y, d) \neq (1, 0, 1)$, then the generalized path $P(x, y, d)$ is a metric two-dimensional graph with the metric basis $B = \{v_{0,d}, v_{d,0}\}$ and the basic distance $d$ with respect to $B$. Moreover, $\langle v_{i,j} | B \rangle = (i, j)$ for each $v_{i,j} \in P$.

**Proof.** At first we note that if $(x, y, d) \neq (1, 0, 1)$, then $P(x, y, d)$ is not a path. We can therefore deduce that $\dim_M(P(x, y, d)) \geq 2$. We show that $B = \{a := v_{0,d}, b := v_{d,0}\}$ is a metric basis for $P(x, y, d)$. In fact, we use induction on $i+j$ to show that $\langle v_{i,j} | B \rangle = (i, j)$ for each $v_{i,j} \in P$.

The minimum possible value for $i+j$ is $d$. There are $d+1$ vertices

$$v_{i,j} = v_{0,d}, v_{1,d-1}, \ldots, v_{d-1,1}, v_{d,0}$$

with $i+j = d$. Consider the shortest path

$$a = v_{0,d}, v_{1,d-1}, \ldots, v_{d-1,1}, v_{d,0} = b$$

to see that $\langle v_{i,j} | B \rangle = (i, j)$ for these vertices. In particular note that $d(a, b) = d$. Thus $BD_B(P(x, y, d)) = d$.

Now let $\langle v_{i,j} | B \rangle = (i, j)$ for each vertex $v_{i,j}$ with $i+j < N$. Let $v_{k,\ell}$ be a vertex with $k+\ell = N$. Any path from $v_{i,j}$ to $a$ should pass from one of the vertices $v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}$. The distance between each of these vertices to $a$ is $i-1$, by the induction hypothesis. Thus $d(v_{i,j}, a) = i$. A similar argument shows that $d(v_{i,j}, b) = j$. \hfill \qed

**Lemma 2.5.** Let $x$ and $d$ be two positive integers with $x \geq d$ and let $y$ be a nonnegative integer. Then $P(x, y, d) = P(x, y, 1) \cup P(x, x, d-1)$ and $P(x, y, 1) \cap P(x, x, d-1)$ is a path.
Proof. Let $V(\mathcal{P}) = \bigcup_{i=0}^{x}P_i$, where $P_i = \{v_{i|i-d|}, v_{i|i-d+1}, \ldots, v_{i+d}\}$ for $0 \leq i \leq y$ and $P_i = \{v_{i|i-d|}, v_{i|i-d+1}, \ldots, v_{i,i+d}\}$ for $y + 1 \leq i \leq x$.

Put

\[ P'_i = \{v'_{i,j-(d-1)}: v_{i,j} \in P_i \text{ and } j \geq i + d - 2\}, \]
\[ P''_i = \{v''_{i-1,j}: v_{i,j} \in P_i \text{ and } j \leq i + d - 2\}. \]

Now if $\mathcal{P}'$ is the subgraph of $\mathcal{P}$ induced by $\bigcup_{i=0}^{x}P'_i$ and $\mathcal{P}''$ is the subgraph of $\mathcal{P}$ induced by $\bigcup_{i=1}^{x}P''_i$, then $\mathcal{P}' = \mathcal{P}(x, y, 1), \mathcal{P}'' = \mathcal{P}(x, x, d - 1), \mathcal{P}(x, y, d) = \mathcal{P}' \cup \mathcal{P}''$ and $\mathcal{P}' \cap \mathcal{P}''$ is the path $\{v_{1,d-1}, v_{2,d}, \ldots, v_{x,x+d-2}\}$.

**Theorem 2.6.** A simple graph $G$ is a metric two-dimensional graph with the basic distance $d$ if and only if it is a subgraph of a generalized path $\mathcal{P}(x, y, d)$ with $(x, y, d) \neq (1, 0, 1)$ satisfying the following properties:

i. $v_{0,d}, v_{d,0} \in G$;

ii. $N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}\} \neq \emptyset$ and $N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i-1,j}, v_{i+1,j-1}\} \neq \emptyset$ for each $v_{ij} \in G$.

Proof. An inductive argument proves that any subgraph of $\mathcal{P}(x, y, d)$ with $(x, y, d) \neq (1, 0, 1)$ possessing the properties (i) and (ii) is a metric two-dimensional graph with the basis $B = \{a := v_{0,d}, b := v_{d,0}\}$ and the basic distance $d$.

Conversely, suppose that $G$ is a metric two-dimensional graph with the basis $B = \{a, b\}$ and the basic distance $d$. Let

\[ x := \max\{d(v, a) : v \in G\}, \]

and

\[ y := \max\{i : (i, i+d) = \langle v \vert B \rangle, \text{ for some } v \in G\}. \]

Define $\varphi : G \rightarrow \mathcal{P}(x, y, d)$ by $\varphi(v) = v_{i,j}$, where $(i, j) = \langle v \vert B \rangle$. We show that

\[ |i - d| \leq j \leq i + d, \quad \text{for } i = 0, \ldots, y, \]
\[ |i - d| \leq j \leq i + d - 1, \quad \text{for } i = y + 1, \ldots, x. \]

We have $d(v, a) = i$ and $d(v, b) = j$, since $(i, j) = \langle v \vert B \rangle$. The triangle inequality implies that $d = d(a, b) \leq d(a, v) + d(v, b) = i + j$. Moreover, $j = d(v, b) \leq d(v, a) + d(a, b) = i + d$ and $i = d(v, a) \leq d(v, b) + d(b, a) = j + d$. Thus $|i - d| \leq j \leq i + d$ for each $0 \leq i \leq x$.

If $i \geq y + 1$, then $j$ cannot be $i + d$, since otherwise we should have $(i, i+d) = \langle v \vert B \rangle$ which contradicts the definition of $y$. Hence $j \leq i + d - 1$ for $i \geq y + 1$. 


A CHARACTERIZATION FOR METRIC TWO-DIMENSIONAL GRAPHS 183

We therefore have \( \varphi(V(G)) \subseteq V(\mathcal{P}(x, y, d)) \). Now let \( e = uv \) be an edge in \( V(G) \). If \( \varphi(u) = v_{i,j} \) and \( \varphi(v) = v_{k\ell} \), then
\[
i = d(u, a) \leq d(u, v) + d(v, a) = 1 + k,
\]
and
\[
k = d(v, a) \leq d(v, u) + d(u, a) = 1 + i.
\]
Thus \(|i - k| \leq 1\). By the same argument, \(|j - \ell| \leq 1\). This shows that \( k = i - 1, i \) or \( i + 1 \) and \( \ell = j - 1, j \) or \( j + 1 \). Whence \( \varphi(e) \) is an edge in \( \mathcal{P}(x, y, d) \) and so \( G \) is a subgraph of \( \mathcal{P}(x, y, d) \).

Clearly, \( v_{0,d} = a, v_{d,0} = b \in G \). To show that \((ii)\) does also hold, note that if, for example, \( N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}\} = \emptyset \), then there is no path with the length \( i \) from \( v_{i,j} \) to \( a \).

\[\Box\]

3. Enumerating of Metric two-dimensional Graphs with the Basic Distance 1

Lemma 2.5 shows that any generalized path \( \mathcal{P}(x, y, d) \) can be regarded as a larger path \( \mathcal{P}(x', y', d') \). Thus the generalized path mentioned in Theorem 2.6 is not unique. A simple argument based on the property \((ii)\) of Theorem 2.6 implies that if \( x = \max\{d(v, a) : v \in G\}, y = \max\{i : (i, i + d) = \langle v|B \rangle \), for some \( v \in G \) and \( d = d(a, b) \), then the boundary \( \partial \mathcal{P}(x, y, d) \)
\[
v_{0,d}, v_{1,d-1}, v_{2,d-2}, \ldots, v_{d,0}, v_{d+1,1}v_{d+2,2}, \ldots, v_{x,x-d}, v_{1,d+1}, v_{2,d+2}, \ldots, v_{y,y+d}
\]
of \( \mathcal{P}(x, y, d) \) are vertices of \( G \). Whence this \( x, y \) and \( d \) are the least possible values such that \( G \) is a subgraph of \( \mathcal{P}(x, y, d) \).

**Definition 3.1.** Let \( G \) be a simple metric two-dimensional graph. We say that \( G \) is fitted in \( \mathcal{P}(x, y, d) \), denoted by \( G \subseteq \mathcal{P}(x, y, d) \), if
\[
x = \max\{d(v, a) : v \in G\},
\]
\[
y = \max\{i : (i, i + d) = \langle v|B \rangle \), for some \( v \in G \},
\]
\[
d = d(a, b),
\]
or equivalently \( G \) contains the boundary \( \partial \mathcal{P}(x, y, d) \)
\[
v_{0,d}, v_{1,d-1}, v_{2,d-2}, \ldots, v_{d,0}, v_{d+1,1}v_{d+2,2}, \ldots, v_{x,x-d}, v_{1,d+1}, v_{2,d+2}, \ldots, v_{y,y+d}
\]
of \( \mathcal{P}(x, y, d) \). The parameters \( x \) and \( y \) are called the length and width of \( G \) and are denoted by \( \ell(G) \) and \( w(G) \), respectively.

We now want to enumerate the number of \( n \) vertex metric two-dimensional graph with the basic distance 1. Prior to this, we enumerate the number of \( n \) vertex metric two-dimensional graph with the length \( x \), width \( y \), and the basic distance 1. We denote the latter number by \( \nu(n; x, y) \).
Lemma 3.2. \( \nu(n; x, y) \geq 1 \) if and only if \( x + y + 2 \leq n \leq 2x + y + 1 \).

Proof. Suppose that there is an \( n \) vertex metric two-dimensional graph \( G \) with the length \( x \), width \( y \) and the basic distance 1. Using Theorem 2.6, we fit it in \( \mathcal{P}(x, y, 1) \). Since the boundary of \( \mathcal{P}(x, y, 1) \) has \( x + y + 1 \) elements, we should have \( n \geq x + y + 1 \). If \( n = x + y + 1 \), then \( G = \partial \mathcal{P}(x, y, 1) \) which is a path and has metrics dimension 1. Thus \( n \geq x + y + 2 \). Moreover, \( n = |V(G)| \leq |V(\mathcal{P}(x, y, 1))| = 2x + y + 1 \).

On the other hand, if \( x + y + 2 \leq n \leq 2x + y + 1 \), then we can write \( n = x + y + 1 + r \), where \( 1 \leq r \leq x \). Now consider the subgraph of \( \mathcal{P}(x, y, 1) \) induced by \( \partial \mathcal{P}(x, y, 1) \cup \{v_1, \ldots, v_r\} \). This is an \( n \) vertex subgraph of \( \mathcal{P}(x, y, 1) \) satisfying (i) and (ii) of Theorem 2.6. \( \square \)

Based on Lemma 3.2, for simplicity, we denote \( \nu(n; x, y) \) by \( \mu(m; x, y) \). We note that \( \mu(m; x, y) \geq 1 \) if and only if \( 1 \leq m \leq x \).

Lemma 3.3. \( \mu(x; x, y) = 4 \times 2^{y-1} \times 10^{x-y} \) for each \( x \geq y \geq 1 \) and \( \mu(x; x, 0) = 2 \times 10^{x-1} \) for each \( x \geq 1 \).

Proof. Let \( G \) be an \( n \) vertex metric two-dimensional graph \( G \) with the length \( x \), width \( y \) and the basic distance 1, where \( n = 2x + y + 1 \). Thus \( G \subseteq \mathcal{P}(x, y, 1) \) and the induced subgraph \( \partial \mathcal{P}(x, y, 1) \) of \( \mathcal{P}(x, y, 1) \) should be a subgraph of \( G \). For other vertices

\[
\{v_{1,1}, \ldots, v_{y,y}, v_{y+1,y+1}, \ldots, v_{x,x}\},
\]

we should put the edges in such a way that (ii) of Theorem 2.6 is satisfied. For \( v_{1,1} \) putting edges \( v_{1,1}v_{0,1} \) and \( v_{1,1}v_{1,0} \) is compulsory, and we have 4 choices for ‘to put’ or ‘not to put’ the edges \( v_{1,1}v_{1,2} \) and \( v_{1,1}v_{2,1} \).

If \( 1 < i \leq y \), then for \( v_{i,j} \) putting one of the 5 sets of edges,

\[
\{v_{i,i}v_{i-1,i-1}\}, \{v_{i,i}v_{i-1,i}, v_{i,i}v_{i-1,i-1}\}, \{v_{i,i}v_{i-1,i-1}, v_{i,i}v_{i-1,i}\},
\]

\[
\{v_{i,i}v_{i-1,i-1}, v_{i,i}v_{i-1,i-1}\}, \{v_{i,i}v_{i-1,i-1}, v_{i,i}v_{i-1,i}v_{i,i}v_{i,i-1}\}
\]

is compulsory and we have 4 choices for ‘to put’ or ‘not to put’ the edges \( v_{i,i}v_{i,i+1} \) and \( v_{i,i}v_{i+1,i} \).

If \( y + 1 \leq i \leq x \), then the 4 choices decreases into 2 choices, since we do not have \( v_{i,i+1} \).

Finally, if \( y = 0 \), then we have 2 choices for \( v_{1,1} \) and 10 choices for \( v_{i,j} \) when \( 1 < i \leq x \). \( \square \)

Though we know that \( \mu(0; x, y) = 0 \), but for the following recursive relation, we need to assume, as a convenient, that \( \mu(0; x, y) = 1 \).
Furthermore, for $y \geq 1$, we assume that

$$
\omega(j) = \begin{cases} 
4, & j = 1, \\
20, & 2 \leq j \leq y, \\
10, & y + 1 \leq j \leq x,
\end{cases}
$$

and for $y = 0$ we assume that

$$
\omega(j) = \begin{cases} 
2, & j = 1, \\
10, & 2 \leq j \leq x.
\end{cases}
$$

**Theorem 3.4.** Let $x$ be a positive integer, let $y$ be a nonnegative integer, and let $1 \leq m < x$. Then $\mu(m; x, y)$ satisfies the recursive relation

$$
\mu(m; x, y) = \sum_{i=1}^{m+1} \prod_{j=1}^{i-1} \omega(j) \cdot \mu(m - (i - 1); x - i, \max\{y - i, 0\}),
$$

with the boundary values

$$
\mu(0; x, y) = 1, \quad \mu(x; x, y) = \prod_{j=1}^{x} \omega(j).
$$

**Proof.** To determine $\mu(m; x, y)$, we in fact need to enumerate the number of $n = x + y + 1 + m$ vertex metric two-dimensional subgraphs $G$ of $P(x, y, 1)$ with the basic distance 1. Let $m < x$. Then there is a vertex $v_{i,i} \in P(x, y, 1) \setminus G$. Let $i$ be the first index such that $v_{i,i} \in P(x, y, 1) \setminus G$. Then $1 \leq i \leq m + 1$. Since $v_{1,1}, \ldots, v_{i-1,i-1} \in G$, we have $\prod_{j=1}^{i-1} \omega(j)$ choices for selecting appropriate edges. Then we have $\mu(m - (i - 1); x - i, \max\{y - i, 0\})$ choices for selecting other edges for other vertices of $G$. \hfill \Box

**Corollary 3.5.** Let $x$ be a positive integer and let $1 \leq m < x$. Then

$$
\mu(m; x, 0) = \mu(m; x - 1, 0) + \sum_{i=2}^{m+1} 2 \times 10^{i-2} \cdot \mu(m - (i - 1); x - i, 0).
$$

**Example 3.6.** We evaluate $\mu(m; x, 0)$ for $m = 1, 2, 3$ and $x > m$.

A simple verification shows that $\mu(1; x, 0) = 2x$. For $m = 2 < x$ we have

$$
\begin{align*}
\mu(2; x, 0) &= \mu(2; x - 1, 0) + 2 \mu(1; x - 2, 0) + 20 \mu(0; x - 3, 0) \\
&= \mu(2; x - 1, 0) + 2 \cdot 2(x - 2) + 20 \\
&= \mu(2; x - 1, 0) + 4(x + 3).
\end{align*}
$$
Iterating the above equation, we have
\[
\mu(2; x, 0) = \mu(2; x - 1, 0) + 4(x + 3)
\]
\[
= \mu(2; x - 2, 0) + 4(x + 2) + 4(x + 3)
\]
\[
= \mu(2; x - 3, 0) + 4(x + 1) + 4(x + 2) + 4(x + 3)
\]
\[
= \ldots
\]
\[
= \mu(2; 2, 0) + 4(2 + 4) + \ldots 4(x + 3)
\]
\[
= 20 + 4 \left( \frac{(x + 3)(x + 4)}{2} - 15 \right)
\]
\[
= 2(x - 1)(x + 8).
\]

Finally, for \( m = 3 < x \), we have
\[
\mu(3; x, 0) = \mu(3; x - 1, 0) + 2\mu(2; x - 2, 0) + 20\mu(1; x - 3, 0)
\]
\[
+ 200\mu(0; x - 4, 0)
\]
\[
= \mu(3; x - 1, 0) + 2 \cdot 2(x - 3)(x + 6) + 20 \cdot 2(x - 3) + 200
\]
\[
= \mu(3; x - 1, 0) + 4(x^2 + 13x + 2).
\]

A similar method gives
\[
\mu(3; x, 0) = \frac{4}{3} x^3 + 28x^2 + \frac{104}{3} x - 192.
\]

**Corollary 3.7.** Let \( x \) be a positive integer and let \( 1 \leq m < x \). Then \( \mu(m; x, 0) \) is a polynomial of \( x \) of degree \( m \).

**Proof.** Using induction on \( m + x \), we can assume that the right hand side of Corollary 3.5 is a polynomial of \( x \) of degree \( m \). Whence the left hand side is also a polynomial of \( x \) of degree \( m \). \( \square \)

We now can simply evaluate \( \nu(n) \); the number of all \( n \) vertex labeled metric two-dimensional graph with the basis \( B = \{a, b\} \) and the basic distance 1.

**Theorem 3.8.** The number of all \( n \) vertex labeled metric two-dimensional graph \( G \) with the basis \( B = \{a, b\} \) and the basic distance 1, is
\[
\nu(n) = \sum_{y=0}^{[\frac{n-1}{3}]} \sum_{x=[\frac{n-y-1}{2}]}^{n-y-2} \mu(n - x - y - 1; x, y).
\]

**Proof.** Each \( G \) can be fitted in a \( \mathcal{P}(x, y, 1) \) where, by Lemma 3.2, we should have \( x + y + 2 \leq n \leq 2x + y + 1 \). Thus the valid values of \( x \) and \( y \) are \( 0 \leq y \leq [\frac{n-1}{3}] \) and \( [\frac{n-y-1}{2}] \leq x \leq n - y - 2 \). We know that the number of metric two-dimensional subgraph of \( \mathcal{P}(x, y, 1) \) is \( \mu(n - x - y - 1; x, y) \). \( \square \)
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