P-CLOSURE IN PSEUDO BCI-ALGEBRAS

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Abstract. In this paper, for any non-empty subset $C$ of a pseudo BCI-algebra $X$, the concept of $p$-closure of $C$, denoted by $C^{pc}$, is introduced and some related properties are investigated. Applying this concept, a characterization of the minimal elements of $X$ is given. It is proved that $C^{pc}$ is the least closed pseudo BCI-ideal of $X$ containing $C$ and $K(X)$ for any ideal $C$ of $X$. Finally, by using the concept of $p$-closure, a closure operator is introduced.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras as a generalization of set-theoretic difference and propositional calculi \[5, 6\]. We refer useful textbooks for BCK/BCI-algebra to \[9, 10\]. The notion of pseudo BCI-algebras was introduced by W.A. Dudek and Y.B. Jun \[4\] in 2008 as an extention of BCI-algebras, and investigated some related properties. Y.B. Jun, et. al. introduced the notion of pseudo BCI-ideals and pseudo BCI-homomorphism, and showed that the pseudo BCK-part of pseudo BCI-algebras is a pseudo BCI-ideal. In \[2\], G. Dymek introduced the notion of p-semisimple pseudo BCI-algebras, and established some necessary and sufficient condition for a pseudo BCI-algebra to be p-semisimple pseudo BCI-algebra. Also, he proved that there is a one to one relationship between p-semisimple pseudo BCI-algebra and groups. In \[8\], Y.H. Kim and K.S. So defined the minimal elements of pseudo BCI-algebras, and showed that the set of all minimal elements of a
pseudo BCI-algebra $X$ forms a subalgebra of $X$. Recently, G. Dymek [1] introduced the notion of period of elements of pseudo BCI-algebras and investigated their properties. It is known that for any non-empty subset $C$ of a BCI-algebra $X$, the generated ideal $\langle C \cup C^\circ \rangle$ is the least closed ideal of $X$ containing $C$, where $C^\circ = \{0 \ast x \mid x \in C\}$ [10]. According to this fact, for any non-empty subset $C$ of a pseudo BCI-algebra $X$, the concept of p-closure of $C$, denoted by $C_{pc}$, is defined as $C_{pc} := \{ x \in X \mid a \ast x \in C \text{ and } a \diamond x \in C \text{ for some } a \in C \}$, and some related properties are investigated. Applying this concept, a characterization of the minimal elements of $X$ is given. A necessary and sufficient condition for a pseudo BCI-algebra to be a p-semisimple BCI-algebra is given. It is proved that $C_{pc}$ is the least closed pseudo BCI-ideal containing $C$ and $K(X)$ for any ideal $C$ of $X$. Finally, by using the concept of $p$-closure, a closure operator is introduced.

2. Preliminary

In this section, we review some definitions and properties that will be used in this paper. For more details, we refer the reader to [9, 4].

An algebra $(X, \ast, 0)$ of type $(2,0)$ is called a $BCI$-algebra if it satisfies the following conditions: for any $x, y, z \in X$,

BCI-1: $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$,

BCI-2: $x \ast 0 = 0$,

BCI-3: $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$.

A $BCI$-algebra $(X, \ast, 0)$ satisfying $0 \ast x = 0$ for all $x \in X$ is called a $BCK$-algebra.

In any $BCI$-algebra (and $BCK$-algebra) $X$, one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \ast y = 0$.

**Definition 2.1.** A pseudo $BCI$-algebra is a structure $\mathfrak{X} = (X, \leq, \ast, \diamond, 0)$, where $\leq$ is a binary relation on set $X$, $\ast$ and $\diamond$ are binary operations on $X$ and 0 is an elements of $X$ satisfying the following axioms: for all $x, y, z \in X$,

- $(a_1) \ (x \ast y) \diamond (x \ast z) \leq z \ast y, \ (x \diamond y) \ast (x \diamond z) \leq z \diamond y$,
- $(a_2) \ x \ast (x \diamond y) \leq y, \ x \diamond (x \ast y) \leq y$,
- $(a_3) \ x \leq x$,
- $(a_4) \ x \leq y, \ y \leq x \implies x = y$,
- $(a_5) \ x \leq y \iff x \ast y = 0 \iff x \diamond y = 0$.

A pseudo $BCI$-algebra $\mathfrak{X} = (X, \leq, \ast, \diamond, 0)$ satisfying $0 \leq x$ for all $x \in X$ is called a pseudo $BCK$-algebra.

It is obvious that every pseudo $BCI$-algebra (resp: pseudo $BCK$-algebra) satisfying $x \ast y = x \diamond y$ for any $x, y \in X$ is a $BCI$-algebra (resp: $BCK$-algebra).
Any pseudo $BCI$-algebra $\mathfrak{X}$ satisfies the following conditions: for any $x, y, z \in X$,

\begin{enumerate}
  \item $x \leq 0 \Rightarrow x = 0$,
  \item $x \leq y \Rightarrow x \ast z \leq y \ast z$, $x \circ z \leq y \circ z$,
  \item $x \leq y \Rightarrow z \ast y \leq z \ast x$, $z \circ y \leq z \circ x$,
  \item $x \leq y$, $y \leq z \Rightarrow x \leq z$,
  \item $(x \ast y) \circ z = (x \circ z) \ast y$,
  \item $x \ast y \leq z \Leftrightarrow x \circ z \leq y$,
  \item $(x \ast y) \ast (z \ast y) \leq x \ast z$, $(x \circ y) \circ (z \circ y) \leq x \circ z$,
  \item $x \ast (x \circ (x \ast y)) = x \ast y$ and $x \circ (x \circ (x \circ y)) = x \circ y$,
  \item $x \ast 0 = x = x \circ 0$,
  \item $x \ast x = 0 = x \circ x$,
  \item $0 \ast (x \circ y) \leq y \circ x$,
  \item $0 \circ (x \ast y) \leq y \ast x$,
  \item $0 \ast x = 0 \circ x$,
  \item $0 \ast (x \ast y) = (0 \ast x) \circ (0 \ast y)$,
  \item $0 \circ (x \circ y) = (0 \circ x) \ast (0 \circ y)$.
\end{enumerate}

For any $BCI$-algebra (and $BCK$-algebra) $X$, using axioms $(a_3)$, $(a_4)$ and property $(p_3)$, the relation order $\leq$ defined by axiom $(a_5)$, that is,

$$(\forall x, y \in X) \; x \leq y \iff x \ast y = 0 \iff x \circ y = 0,$$

is a partial order.

A non-empty subset $S$ of a pseudo $BCI$-algebra $\mathfrak{X}$ is called a subalgebra of $\mathfrak{X}$ if $x \ast y \in S$ and $x \circ y \in S$ for all $x, y \in S$. It is easily seen that the set $K(\mathfrak{X}) = \{x \in X \mid 0 \leq x\}$ is a subalgebra of $\mathfrak{X}$ (called the maximal pseudo $BCK$-algebra of $\mathfrak{X}$). Then $(K(\mathfrak{X}), \leq, \ast, \circ, 0)$ is a pseudo $BCK$-algebra and so a pseudo $BCI$-algebra $\mathfrak{X}$ is a pseudo $BCK$-algebra if and only if $X = K(\mathfrak{X})$.

An element $a$ of a pseudo $BCI$-algebra $X$ is called minimal if for any $x \in X$ the following holds:

$$a \leq x \implies a = x.$$

We will denote by $M(\mathfrak{X})$ the set of all minimal elements of $X$. Obviously, $0 \in M(\mathfrak{X})$. In [6], it has proved that $a \in X$ is minimal if and only if $a = 0 \ast (0 \circ a)$ if and only if $a = 0 \ast x$ for some $x \in X$. Therefore $M(\mathfrak{X}) = \{x \in X \mid x = 0 \circ (0 \ast x)\} = \{0 \ast x \mid x \in X\}$. A pseudo $BCI$-algebra $\mathfrak{X}$ is called $p$-semisimple if any element of $X$ is minimal. It is easily to seen that $K(\mathfrak{X}) \cap M(\mathfrak{X}) = \{0\}$.

**Proposition 2.2.** [2] Let $\mathfrak{X}$ be a pseudo $BCI$-algebra. Then for any $x, y \in X$ the following are equivalent:

(i) $\mathfrak{X}$ is a $p$-semisimple,
(ii) \( x \ast (x \circ y) = y = x \circ (x \ast y) \),
(iii) \( 0 \ast (0 \circ x) = x = 0 \circ (0 \ast x) \).

For any minimal element \( a \in X \), the branch of \( a \) is defined by \( V(a) := \{ x \in X \mid x \succeq a \} \). Obviously, \( a \in V(a) \) and hence \( V(a) \neq \emptyset \).

Let \( X \) be a pseudo-BCI-algebra. For any none-empty subset \( J \) of \( X \) and any element \( y \in X \) we denote
\[ *(y, J) := \{ x \in X \mid x \ast y \in J \} \quad \text{and} \quad \diamond (y, J) := \{ x \in X \mid x \circ y \in J \} .\]

**Definition 2.3.** [7] A subset \( J \) of a pseudo BCI-algebra \( X \) is called a pseudo BCI-ideal of \( X \) if

(I1) \( 0 \in J \),
(I2) \( (\forall y \in J) \ (*(y, J) \subseteq J \) and \( \diamond (y, J) \subseteq J \)\)

**Theorem 2.4.** [7] If \( J \) is a pseudo BCI-ideal of a pseudo BCI-algebra \( X \), then the following hold: for any \( x, y, z \in X \),

(i) \( x \in J \) and \( y \preceq x \implies y \in J \),
(ii) \( y \in J \) and \( z \ast y \in J \implies z \in J \),
(iii) \( y \in J \) and \( z \circ y \in J \implies z \in J \).

A pseudo BCI-ideal \( J \) of a pseudo BCI-algebra \( X \) is called closed if \( J \) is closed under operations * and \( \circ \). A pseudo BCI-ideal \( J \) of a pseudo BCI-algebra \( X \) is closed if and only if \( 0 \ast x = 0 \circ x \in J \) for any \( x \in J \) (see [7]).

### 3. Main results

In this section, we start by introducing the concept of \( p \)-closure for a non-empty subset \( C \) of a pseudo BCI-algebra \( X \), and then investigate some related properties.

In what follows, let \( X \) denote a pseudo BCI-algebra unless otherwise specified.

**Definition 3.1.** For any non-empty subset \( C \) of \( X \), we define the \( p \)-closure of \( C \) by the set
\[ C^{pc} := \{ x \in X \mid a \ast x \in C \text{ and } a \circ x \in C \text{ for some } a \in C \} .\]

Obviously, \( 0 \in C^{pc} \).

The following lemma is an immediate consequence from Definition 3.1 and \((p_9)\).

**Lemma 3.2.** For any non-empty subsets \( C \) and \( D \) of \( X \), the following holds:

(i) if \( C \subseteq D \), then \( C^{pc} \subseteq D^{pc} \),
(ii) if \( 0 \in C \), then \( C \subseteq C^{pc} \).
In the following theorem, we give a characterization of the minimal elements of \( X \).

**Theorem 3.3.** An element \( a \) of \( X \) is minimal if and only if \( \{a\}^{pc} = \mathcal{K}(\mathfrak{X}) \).

**Proof.** (\( \Rightarrow \)) Let \( a \) be a minimal element of \( X \). Assume that \( x \in \{a\}^{pc} \). Then \( a \odot x = a \triangleleft x \) and so, using \((p_5)\), we have \( 0 = (a \triangleleft x) \ominus a = (a \ominus a) \odot x = 0 \odot x \). It follows that \( x \in \mathcal{K}(\mathfrak{X}) \). Hence \( \{a\}^{pc} \subseteq \mathcal{K}(\mathfrak{X}) \). To prove the reverse inclusion, let \( x \in \mathcal{K}(\mathfrak{X}) \). Then \( 0 \triangleright x = 0 \), and so we have

\[
 a \triangleright x = (0 \triangleright (0 \triangleright a))(x)
\]

by the minimality of \( a \),

\[
 = (0 \triangleright x) \ominus (0 \triangleright a)
\]

by \((p_5)\),

\[
 = 0 \ominus (0 \triangleright a)
\]

by \((p_{13})\),

\[
 = a,
\]

by the minimality of \( a \) that is, \( a \triangleright x = a \), which implies that \( x \in \{a\}^{pc} \). Therefore \( \mathcal{K}(\mathfrak{X}) \subseteq \{a\}^{pc} \) and so \( \{a\}^{pc} = \mathcal{K}(\mathfrak{X}) \).

(\( \Leftarrow \)) Assume that \( \{a\}^{pc} = \mathcal{K}(\mathfrak{X}) \). Let \( b \in X \) with \( b \preceq a \). Then \( 0 \preceq a \odot b \) and so \( a \odot b \in \mathcal{K}(\mathfrak{X}) \). Thus \( a \odot b \in \{a\}^{pc} \) and hence \( a \odot (a \odot b) = a \).

It follows from \((p_5)\) that \( a \odot b = (a \odot (a \odot b)) \triangleright b = (a \odot b) \ominus (a \odot b) = 0 \), that is, \( a \preceq b \). Hence \( a = b \). Therefore \( a \) is a minimal element of \( X \). \( \square \)

In the following theorem, we give a necessary and sufficient condition for a pseudo \( BCI \)-algebra to be a pseudo \( BCK \)-algebra.

**Theorem 3.4.** \( \mathfrak{X} \) is a pseudo \( BCK \)-algebra if and only if \( \{0\}^{pc} = X \).

**Proof.** (\( \Rightarrow \)) Let \( \mathfrak{X} \) be a pseudo \( BCK \)-algebra. Then for any \( x \in X \), \( 0 \triangleleft x = 0 \odot x \). It follows that \( x \in \{0\}^{pc} \) for any \( x \in X \). Therefore \( \{0\}^{pc} = X \).

(\( \Leftarrow \)) Assume that \( \{0\}^{pc} = X \). Then using Theorem 3.3, we get \( X = \mathcal{K}(\mathfrak{X}) \). This implies that \( \mathfrak{X} \) is a pseudo \( BCK \)-algebra. \( \square \)

**Corollary 3.5.** \( \mathfrak{X} \) is a pseudo \( BCK \)-algebra if and only if \( C^{pc} = X \) for any subset \( C \) of \( X \) containing \( 0 \).

**Proof.** Using Lemma 3.2(i) and Theorem 3.4, the proof is straightforward. \( \square \)

In the following, we introduce some subsets of \( X \) whose \( p \)-closure are maximal pseudo \( BCK \)-algebra of \( \mathfrak{X} \).

**Theorem 3.6.** For any \( \mathfrak{X} \), the following hold:

(i) if \( C \) is a subset of \( \mathcal{K}(\mathfrak{X}) \) and \( 0 \in C \), then \( C^{pc} = \mathcal{K}(\mathfrak{X}) \),

(ii) \( \mathcal{K}(\mathfrak{X})^{pc} = \mathcal{K}(\mathfrak{X}) \),
(iii) for any element $c$ of $X$, $\{A(c)\}^{pc} = K(\mathfrak{X})$, where $A(c) = \{x \in X \mid x \preceq c\}$.

Proof. (i) Since $\{0\} \subseteq C \subseteq K(\mathfrak{X})$, it follows from Lemma 3.2(i) that $\{0\}^{pc} \subseteq C^{pc} \subseteq K(\mathfrak{X})^{pc}$. Thus by Theorem 3.3, we obtain $K(\mathfrak{X}) \subseteq C^{pc} \subseteq K(\mathfrak{X})$, which implies that $C^{pc} = K(\mathfrak{X})$.

(ii) It is an immediate consequence of (i).

(iii) Let $x \in K(\mathfrak{X})$. Then $0 \ast x = 0 = 0 \diamond x$ and so $(c \ast x) \diamond c = (c \diamond c) \ast x = 0 \ast x = 0$. This implies that $c \ast x \preceq c$ and so $c \ast x \in A(c)$. Moreover, $c \in A(c)$. Hence, $x \in A(c)^{pc}$ and so $K(\mathfrak{X}) \subseteq A(c)^{pc}$. Now let $x \in A(c)^{pc}$. Then there exists $t \in A(c)$ such that $t \ast x \preceq c$, that is, $(t \ast c) \ast x = 0$. On the other hand, from $t \in A(c)$ we have $t \ast c = 0$. Thus, $0 \ast x = 0$ and so $x \in K(\mathfrak{X})$. Therefore $A(c)^{pc} = K(\mathfrak{X})$. □

Proposition 3.7. For any subset $C$ of $X$ containing $M(\mathfrak{X})$, $C^{pc} = X$.

Proof. (i) Let $x \in X$. We know that $0 \ast (0 \ast x)$ is a minimal element of $\mathfrak{X}$, and so $0 \ast (0 \ast x) \in M(\mathfrak{X})$. Thus, $0 \ast (0 \ast x) \in C$. Now, using $(p_5)$, we get $(0 \ast (0 \ast x)) \ast x = 0 \in C$ and $(0 \ast (0 \ast x)) \diamond x = 0 \in C$, which implies $x \in C^{pc}$. Therefore $C^{pc} = X$. □

Lemma 3.8. Let $C$ be a subalgebra of $X$. Then the following statement are equivalent: for any $x \in X$,

(i) $x \in C^{pc}$.
(ii) $0 \ast x \in C$.
(iii) $0 \ast x \in C^{pc}$.

Proof. (i)$\Rightarrow$(ii) Let $x \in C^{pc}$. Then $a \ast x \in C$ and $a \diamond x \in C$ for some $a \in C$, and so, since $C$ is closed, we get $(a \ast x) \diamond a \in C$. On the other hand, we have $(a \ast x) \diamond a = (a \diamond a) \ast x = 0 \ast x$. Therefore $0 \ast x \in C$.

(ii)$\Rightarrow$(iii) This is obvious by Lemma 3.2(ii).

(iii)$\Rightarrow$(i) Let $0 \ast x \in C^{pc}$. Then there exists $a \in C$ such that $a \ast (0 \ast x) \in C$ and $a \diamond (0 \ast x) \in C$. Since $C$ is closed, we obtain $(a \ast (0 \ast x)) \diamond a \in C$. But using $(p_5)$, we have $(a \ast (0 \ast x)) \diamond a = 0 \ast (0 \ast x)$. Hence $0 \ast (0 \ast x) \in C$. Now, by $(p_5)$, we get $(0 \ast (0 \ast x)) \ast x = 0 = (0 \ast (0 \ast x)) \diamond x$. Therefore, it follows from $0 \in C$ that $x \in C^{pc}$. □

The following follows from Lemma 3.8.

Corollary 3.9. If $C$ is a subalgebra of $X$, then so is $C^{pc}$.

In the following theorem, for any subalgebra $C$ of $X$, we give a characterization of $C^{pc}$ by some branches of $C$.

Theorem 3.10. If $C$ is a subalgebra of $X$, then $C^{pc} = \bigcup_{c \in C} V(0 \ast c)$. 
Proof. Let $x \in C^{pc}$. Then by Lemma 3.8, $0 \ast x \in C$. Since $0 \ast (0 \ast x) \preceq x$, by putting $c = 0 \ast x$, we get $x \in V(0 \ast c)$. This implies that $C^{pc} \subseteq \bigcup_{c \in C} V(0 \ast c)$. In order to show the reverse inclusion, let $x \in \bigcup_{c \in C} V(0 \ast c)$. Then there exists $c \in C$ such that $x \in V(0 \ast c)$. Thus, $0 \ast c \preceq x$ and so $(0 \ast c) \ast x = 0$ and $(0 \ast c) \diamond x = 0$. Moreover, since $C$ is a subalgebra, we have $0 \ast c \in C$ and hence $x \in C^{pc}$. Therefore $\bigcup_{c \in C} V(0 \ast c) \subseteq C^{pc}$, and so the proof is completed. \hfill $\Box$

In the following, we establish an important property of the $p$-closure.

**Theorem 3.11.** If $C$ is a pseudo $BCI$-ideal of $\mathcal{X}$, then $C^{pc}$ is a pseudo $BCI$-ideal of $\mathcal{X}$, too.

**Proof.** We first prove that $C^{pc}$ is a pseudo $BCI$-ideal of $\mathcal{X}$. Clearly, $0 \in C^{pc}$. Now, we show that $\ast (y, C^{pc}) \subseteq C^{pc}$ and $\diamond (y, C^{pc}) \subseteq C^{pc}$ for any $y \in C^{pc}$. Let $x \in \ast (y, C^{pc})$. Then $x \ast y \in C^{pc}$, and so there exists $b \in C$ such that $b \ast (x \ast y) \in C$ and $b \diamond (x \ast y) \in C$. Also, from $y \in C^{pc}$, we have $a \ast y \in C$ and $a \diamond y \in C$ for some $a \in C$. We first show that $b \diamond (0 \ast a) \in C$. It is easy to see that $(b \diamond (0 \ast a)) \ast b = (b \ast b) \diamond (0 \ast a) = 0 \diamond (0 \ast a) \leq a$. Thus, since $a, b \in C$, we conclude

$$b \diamond (0 \ast a) \in C. \tag{3.1}$$

Now, we show that $x \in C^{pc}$. For this purpose, using $(p_5)$ and axiom $(a1)$, we have

$$((b \diamond (0 \ast a)) \diamond x) \ast (b \diamond (x \ast y)) = ((b \diamond (0 \ast a)) \ast (b \diamond (x \ast y))) \diamond x \leq ((x \ast y) \diamond (0 \ast a)) \diamond x$$

$$= ((x \diamond (0 \ast a)) \ast y) \diamond x$$

$$= ((x \diamond (0 \ast a)) \diamond x) \ast y$$

Thus

$$((b \diamond (0 \ast a)) \diamond x) \ast (b \diamond (x \ast y)) \leq ((x \diamond (0 \ast a)) \diamond x) \ast y. \tag{3.2}$$

On the other hand, using $(p_5)$ and axiom $(a1)$ again, we have

$$((x \diamond (0 \ast a)) \diamond x) \ast y \ast (a \ast y) \leq ((x \diamond (0 \ast a)) \diamond x) \ast a$$

$$= ((x \diamond (0 \ast a)) \ast a) \diamond x$$

$$= ((x \ast a) \diamond (0 \ast a)) \diamond x$$

$$\leq (x \ast 0) \diamond x$$

$$= 0.$$

This implies that

$$(x \diamond (0 \ast a)) \ast y \leq a \ast y \tag{3.3}$$
Combining (3.2) and (3.3), we obtain 

\[(b \odot ((0 \ast a) \odot x) \ast (b \odot (x \ast y)) \preceq a \ast y \in C). \]

Thus, since \(b \odot (x \ast y) \in C\), we get \((b \odot (0 \ast a)) \odot x \in C\). Similarly, applying \(a \odot (x \ast y) \in C\) and \(a \odot y \in C\), we can show that \((b \odot (0 \ast a)) \odot x \in C\). Hence, by (3.1), we have \(x \in C^{\text{pc}}\), and so \(*(y, C^{\text{pc}}) \subseteq C^{\text{pc}}\). By the similar argument, we can show that \(\odot (y, C^{\text{pc}}) \subseteq C^{\text{pc}}\). Therefore \(C^{\text{pc}}\) is a pseudo BCI-ideal of \(X\).

The following is another important property of the p-closure.

**Theorem 3.12.** If \(C\) is a pseudo BCI-ideal of \(X\), then \(C^{\text{pc}}\) is a closed pseudo BCI-ideal of \(X\) containing \(K(\mathfrak{X})\).

**Proof.** Let \(x \in C^{\text{pc}}\). Then \(a \ast x \in C\) and \(a \odot x \in C\) for some \(a \in C\). Using \((p_5)\), we get \((a \ast (0 \ast a)) \odot a = 0 \ast (0 \ast a) \preceq a \in C\), and so \(a \ast (0 \ast a) \in C\). Similarly, we have \(a \odot (0 \ast a) \in C\). Thus \(0 \ast a \in C^{\text{pc}}\).

Now, since \((0 \ast x) \ast (a \ast x) \preceq 0 \ast a \in C^{\text{pc}}\), it follows from \(a \ast x \in C \subseteq C^{\text{pc}}\) that \(0 \ast x \in C^{\text{pc}}\). Therefore \(C^{\text{pc}}\) is closed. Also, using Theorem 3.3 and Lemma 3.2, we get \(K(\mathfrak{X}) = \{0\}^{\text{pc}} \subseteq C^{\text{pc}}\), and so the proof is completed.

**Lemma 3.13.** For any \(X\),

\[
K(\mathfrak{X}) = \{x \odot (0 \ast (0 \ast x)) \mid \text{for some } x \in X\}
\]

\[
= \{x \ast (0 \ast (0 \ast x)) \mid \text{for some } x \in X\}.
\]

**Proof.** (i) For any \(x \in X\), we have

\[
0 \ast (x \odot (0 \ast (0 \ast x))) = (0 \ast x) \odot (0 \ast (0 \ast x)) \text{ by } (p_{14})
\]

\[
= (0 \ast x) \odot (0 \ast x) \text{ by } (p_8)
\]

\[
= 0
\]

Thus for any \(x \in X\), \(x \odot (0 \ast (0 \ast x)) \in K(\mathfrak{X})\). Therefore \(\{x \odot (0 \ast (0 \ast x)) \mid \text{for some } x \in X\} \subseteq K(\mathfrak{X})\). On the other hand, if \(x \in K(\mathfrak{X})\), then \(0 \ast x = 0\) and so \(x = x \odot (0 \ast (0 \ast x))\). This implies \(K(\mathfrak{X}) \subseteq \{x \odot (0 \ast (0 \ast x)) \mid \text{for some } x \in X\}\). Therefore \(K(\mathfrak{X}) = \{x \odot (0 \ast (0 \ast x)) \mid \text{for some } x \in X\}\). Similarly, we can show the second part of the lemma.

In the following, we introduce an interesting property of the p-closure.

**Theorem 3.14.** If \(C\) is a pseudo BCI-ideal of \(X\), then \(C^{\text{pc}} = (C^{\text{pc}})^{\text{pc}}\).

**Proof.** Since \(0 \in C^{\text{pc}}\), it follows from Lemma 3.2(ii) that \(C^{\text{pc}} \subseteq (C^{\text{pc}})^{\text{pc}}\). To show the reverse inclusion, let \(x \in (C^{\text{pc}})^{\text{pc}}\). By Theorem 3.12, \(C^{\text{pc}}\)
is a subalgebra of \( \mathfrak{X} \) and so by Lemma 3.8, we get \( 0 \ast x \in C^{pc} \). Then, since \( C^{pc} \) is closed, we have

\[
0 \ast (0 \ast x) \in C^{pc}. \tag{3.4}
\]

By Lemma 3.13, we have \( x \circ (0 \ast (0 \ast x)) \in K(\mathfrak{X}) \). On the other hand, \( K(\mathfrak{X}) \subseteq C^{pc} \). Hence \( x \circ (0 \ast (0 \ast x)) \in C^{pc} \), and so by (3.4), we get \( x \in C^{pc} \). Therefore \( (C^{pc})^{pc} \subseteq C^{pc} \), which completes the proof. □

**Corollary 3.15.** For any \( \mathfrak{X} \), the mapping \( \text{pc} : \mathbb{I}(\mathfrak{X}) \to \mathbb{I}(\mathfrak{X}) \) defined by \( \text{pc}(C) = C^{pc} \) for any \( C \in \mathbb{I}(\mathfrak{X}) \) is a closure operator on \( (\mathbb{I}(\mathfrak{X}), \subseteq) \), where \( \mathbb{I}(\mathfrak{X}) \) denotes the set of all pseudo BCI-ideals of \( \mathfrak{X} \).

**Proof.** It is an immediate consequence from Lemma 3.2 and Theorem 3.14. □

In the following theorem, we give a necessary and sufficient condition for a pseudo BCI-ideal to be closed.

**Theorem 3.16.** Let \( C \) be a pseudo BCI-ideal of \( \mathfrak{X} \). If we denote \( C_0 = \{ x \in C \mid 0 \ast x \in C \} \), then the following are equivalent:

(i) \( C \) is closed,
(ii) \( C = C_0 \),
(iii) \( C^{pc} = C_0^{pc} \).

**Proof.** The proof of (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are easy.

(iii) \( \Rightarrow \) (i) Assume that \( C_0^{pc} = C^{pc} \) and \( x \in C \). Then, by the closeness of \( C^{pc} \), we have \( 0 \ast x \in C^{pc} \) and so by assumption, \( 0 \ast x \in C_0^{pc} \). Thus there exists \( a \in C_0 \) such that \( a \ast (0 \ast x) \in C_0 \) and \( a \circ (0 \ast x) \in C_0 \). From this and definition of \( C_0 \) it follows that \( 0 \ast (a \ast (0 \ast x)) \in C \). Now we have

\[
(0 \ast x) \ast a = (0 \circ (0 \ast (0 \circ x))) \ast a \quad \text{by (p8)}
\]

\[
= (0 \ast a) \circ (0 \ast (0 \ast x)) \quad \text{by axiom (a2)}
\]

\[
= 0 \ast (a \ast (0 \ast x)) \quad \text{by (p14)}
\]

Hence \( (0 \ast x) \ast a \in C \) and so from \( a \in C_0 \subseteq C \), we conclude \( 0 \ast x \in C \). Therefore \( C \) is closed. □

In the following, we consider the \( p \)-closure of intersection of a family of closed pseudo BCI-ideals of \( \mathfrak{X} \).

**Theorem 3.17.** For every family \( \{ C_\alpha \}_{\alpha \in I} \) of closed pseudo BCI-ideals of \( \mathfrak{X} \), \( (\bigcap_{\alpha \in I} C_\alpha)^{pc} = \bigcap_{\alpha \in I} C_\alpha^{pc} \).

**Proof.** By Lemma 3.2(i), \( (\bigcap_{\alpha \in I} C_\alpha)^{pc} \subseteq C_\alpha^{pc} \) for every \( \alpha \in I \). Thus \( (\bigcap_{\alpha \in I} C_\alpha)^{pc} \subseteq \bigcap_{\alpha \in I} C_\alpha^{pc} \). Now let \( x \in \bigcap_{\alpha \in I} C_\alpha^{pc} \). Then for every \( \alpha \in I \), there exists \( c_\alpha \in C_\alpha \) such that \( c_\alpha \ast x \in C_\alpha \). Using (p7) and the
fact that $C_\alpha$ is closed, we conclude $(0\ast x) \ast (c_\alpha \ast x) \leq 0 \ast c_\alpha \in C_\alpha$. Then, it follows from $c_\alpha \ast x \in C_\alpha$ that $0 \ast x \in C_\alpha$ and so $0 \ast x \in \bigcap_{\alpha \in I} C_\alpha$. Also, obviously, $0 \ast x \in \bigcap_{\alpha \in I} C_\alpha$. Thus $x \in (\bigcap_{\alpha \in I} C_\alpha)^{pr}$, and consequently $\bigcap_{\alpha \in I}(C_\alpha)^{pr} \subseteq (\bigcap_{\alpha \in I} C_\alpha)^{pr}$. Therefore $(\bigcap_{\alpha \in I} C_\alpha)^{pr} = \bigcap_{\alpha \in I} C_\alpha^{pr}$. □

To give a characterization of the $p$-semisimple pseudo $BCI$-algebras, we recall the following notation [10].

For any non-empty subset $C$ of $\mathfrak{X}$, we denote
$$C^o := \{0 \ast x \mid x \in C\} = \{0 \ast x \mid x \in C\}.$$ 

**Lemma 3.18.** For any pseudo $BCI$-ideal $C$ of $\mathfrak{X}$, the following hold:

(i) $C^o \subseteq C^{pr}$,

(ii) $(C \cup C^o)^{pr} = C^{pr}$.

**Proof.** (i) Let $0 \ast x \in C^o$ for some $x \in C$. Then, from $0 \ast (0 \ast x) \leq x$, we get $0 \ast (0 \ast x) \in C$. Also, obviously, $0 \ast (0 \ast x) \in C$. Then, $0 \ast x \in C^{pr}$ and so $C^o \subseteq C^{pr}$.

(ii) By (i) and Lemma 3.2(ii), we have $C, C^o \subseteq C^{pr}$. Since $C^{pr}$ is a pseudo $BCI$-ideal, we obtain $C \subseteq (C \cup C^o) \subseteq C^{pr}$, hence $C^{pr} \subseteq (C \cup C^o)^{pr} \subseteq (C^{pr})^{pr}$. Thus by Theorem 3.14, we conclude $(C \cup C^o)^{pr} = C^{pr}$. □

In the next theorem, we give a characterization of the $p$-semisimple pseudo $BCI$-algebras.

**Theorem 3.19.** $\mathfrak{X}$ is $p$-semisimple $\iff (C \cup C^o)^{pr} = C^{pr}$ for all pseudo $BCI$-ideal $C$ of $\mathfrak{X}$.

**Proof.** ($\Rightarrow$) This is obvious by Lemma 3.18(ii).

($\Leftarrow$) Assume that $(C \cup C^o)^{pr} = C^{pr}$ for any pseudo $BCI$-ideal $C$ of $\mathfrak{X}$. Taking $C := \{0\}$, we get $C^o = \{0\}$ and so by Theorem 3.6(ii), we have $C^{pr} = K(\mathfrak{X})$. On the other hand, by assumption, we obtain $C^{pr} = C \cup C^o \geq \{0\}$. Therefore $K(\mathfrak{X}) = \{0\}$ and so by Lemma 3.13, we obtain $x \ast (0 \ast (0 \ast x)) = 0$ for any $x \in X$. On the other hand, $(0 \ast (0 \ast x)) \ast x = 0$. Therefore $0 \ast (0 \ast x) = x$ and so by Proposition 2.2, $\mathfrak{X}$ is a $p$-semisimple $BCI$-algebra. □

In the following theorem, we establish the main result of this paper.

**Theorem 3.20.** For any pseudo $BCI$-ideal $C$ of $\mathfrak{X}$, $C^{pr}$ is the least closed pseudo $BCI$-ideal of $\mathfrak{X}$ containing $C$ and $K(\mathfrak{X})$.

**Proof.** Combining Lemma 3.2(ii) and Theorems 3.11 and 3.12, we conclude $C^{pr}$ is a closed pseudo $BCI$-ideal of $X$ containing $C$ and $K(\mathfrak{X})$. To complete the proof, let $D$ be another closed pseudo $BCI$-ideal of $\mathfrak{X}$ containing $C$ and $K(\mathfrak{X})$, and let $x \in C^{pr}$. Then, since $C^{pr}$ is closed,
we get $0 \star x \in C^{pc}$. But from $C \subseteq D$, we have $C^{pc} \subseteq D^{pc}$. Thus $0 \star x \in D^{pc}$ and so it follows from Lemma 3.8 that $0 \star (0 \star x) \in D$. We note that $x \diamond (0 \star (0 \star x)) \in K(X)$ and so from $K(X) \subseteq D$, we obtain $x \diamond (0 \star (0 \star x)) \in D$. Hence, since $0 \star (0 \star x) \in D$, we conclude $x \in D$. Therefore $C^{pc} \subseteq D$, and so the proof is completed.

\[ \square \]

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References


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P-CLOSURE IN PSEUDO BCI-ALGEBRAS

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بحث در شیء-جبرها

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چکیده مقاله: در این مقاله، برای هر زمینه‌جمعیتی Nاتخی C از یک شیء BCI-جبر X، مفهوم
p-بستار با نامیش C، معروفی شده است و برخی خواص مرتبط با آن مورد بررسی قرار گرفته است.
این BCI-جبرگی این مفهوم توصیفی از عناصر متنی از X ارائه گردیده است. این BCI-جبرگی، شیء
p-کوچکترین شیء ایدآل BCI-جبرها است. در نهایت، با BCI-جبرگی مفهوم
p-بستار، یک عملکرد بستار بیان شده است.

کلمات کلیدی: p-بستار، شیء BCI-جبر، شیء BCI-بیان BCI-بیان BCI-بیان

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