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ON THE ARAKI–LIEB–THIRRING INEQUALITY IN THE SEMIFINITE VON NEUMANN ALGEBRA

YAZHOU HAN

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ABSTRACT. This paper extends a recent matrix trace inequality of Bourin–Lee to semifinite von Neumann algebras. This provides a generalization of the Lieb–Thirring-type inequality in von Neumann algebras due to Kosaki. Some new inequalities, even in the matrix case, are also given for the Heinz means.

1. INTRODUCTION

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices. The Lieb–Thirring inequality [17] states that, for $0 \le A, B \in \mathbb{M}_n$ and $p \ge 1$,

$$\operatorname{Tr}\big((B^{\frac{1}{2}}AB^{\frac{1}{2}})^p\big) \le \operatorname{Tr}(A^pB^p).$$

Let A and B be positive self-adjoint operators on a Hilbert space, and let f be any increasing continuous function on $[0,\infty)$ such that $f(0)=0$ and $t \to f(e^t)$ is a convex function. Araki [2] shows a refinement of the Lieb–Thirring inequality as follows:

$$\operatorname{Tr} f\big((B^{\frac{1}{2}}AB^{\frac{1}{2}})^q\big) \le \operatorname{Tr} f(B^{\frac{q}{2}}A^qB^{\frac{q}{2}}) \quad \text{for all } q \ge 1. \tag{1.1}$$

Here the condition $f(0)=0$ ensures that the trace is well defined; that is, $\infty - \infty$ does not occur. Recently, Bourin and Lee [6] proved that if f is increasing and $t \to f(e^t)$ is convex, then the inequality

$$\operatorname{Tr} f\big((BZ^*AZB)^q\big) \le \operatorname{Tr} f(B^qZ^*A^qZB^q) \quad \text{for all } q \ge 1 \tag{1.2}$$

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