Block-Coppels chaos in set-valued discrete systems

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Abstract
Let \((X, d)\) be a compact metric space and \(f : X \to X\) be a continuous map. Consider the metric space \((K(X), H)\) of all nonempty compact subsets of \(X\) endowed with the Hausdorff metric induced by \(d\). Let \(f : K(X) \to K(X)\) be defined by \(f(A) = \{f(a) : a \in A\}\). We show that Block-Coppels chaos in \(f\) implies Block-Coppels chaos in \(\overline{f}\) if \(f\) is a bijection.

Keywords: Chaos; Discrete system; Dynamical system.

1 Introduction

Let \((X, d)\) be a compact metric space with metric \(d\) and \(f : X \to X\) be a continuous map. For every positive integer \(n\), we define \(f^n\) inductively by \(f^n = f \circ f^{n-1}\), where \(f^0\) is the identity map on \(X\). A map \(f\) is called to be Block-Coppels chaotic [3] if there exist disjoint nonempty compact subsets \(J, K\) of \(X\) and a positive integer \(n\) such that \(J \cup K \subseteq f^n(J) \cap f^n(K)\).

Roman-Flores and Chalco-Cano investigated Robinsons chaos in set-valued discrete systems [5]. Gu investigated Katos chaos in set-valued discrete systems [2]. Devaneys chaos in set-valued discrete systems has been studied in several papers. For example see [4], [1].

In this paper, we investigate the relationships between Block-Coppels chaoticity of \((X, f)\) and Block-Coppels chaoticity of \((K(X), \overline{f})\).

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2 Preliminaries

Let $(X, d)$ be a compact metric space with metric $d$. The distance of a point $x$ from a set $A$ in $X$ is defined by $d(x, A) = \inf \{d(x, a) : a \in A\}$ if $A \neq \emptyset$, and $d(x, \emptyset) = 1$. Let $K(X)$ be the family of all non-empty compact subsets of $X$.

The Hausdorff metric on $K(X)$ is defined by $H(A, B) = \max \{\sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\}\}$ for $A, B \in K(X)$. It is easy to see that $(K(X), H)$ is a compact metric space.

Let $\tau_d$ be the topology of $X$ induced by the metric $d$. The topology $\tau_H$ of $K(X)$ induced by the Hausdorff metric $H$ coincides with the topology $\tau_v$ generated by the basis $\beta_v$ consisting of all sets of the form $G_{0}^{u} \cap G_{1}^{l} \cap \cdots \cap G_{k}^{l}$ where $G_{0}, G_{1}, \ldots, G_{k} \in \tau_{d}, G_{0}^{u} = \{A \in K(X) : A \subseteq G_{0}\}$ and $G_{i}^{l} = \{A \in K(X) : A \cap G \neq \emptyset\}, i = 1, 2, \ldots, k$.

The topology $\tau_v$ is also called the Vietoris topology or the exponential topology on $K(X)$.

If $f : X \in K(X)$ is a continuous map then one can define a continuous map $\bar{f} : K(X) \rightarrow K(X)$ by letting $\bar{f}(A) = \{f(a) : a \in A\}$ for every $A \in K(X)$.

3 Block-Coppels Chaoticity

In this section, we show that Block-Coppels chaoticity of $(X, f)$ implies Block-Coppels chaoticity of $(K(X), f)$ if $f$ is bijection.

**Definition 3.1.** Let $A$ be a subset of $X$, the extension of $A$ to $K(X)$ is defined by $e(A) = \{K \in K(X) : K \subseteq A\}$.

Remark. It is clear that $e(A) = \emptyset$ if and only if $A = \emptyset$.

**Lemma 3.1.** Let $A$ be a non-empty compact subset of $X$. Then, $e(A)$ is a non-empty compact subset of $K(X)$.

**Proof.** It is sufficient to show that $(e(A))^c$ is open because in this case $e(A)$ is closed and a closed subset of a compact space, is compact. If $K \in (e(A))^c$ then $K \not\subseteq A$. Therefore $K \cap A^c \neq \emptyset$, and hence $K \in (A^c)^c$. So that

$$ (e(A))^c \subseteq (A^c)^l $$  \hspace{1cm} (1)

On the other hand if $K \in (A^c)^l$ then $K \cap A^c \neq \emptyset$, therefore $K \not\subseteq A$ and hence $K \not\in e(A)$. So that $K \in (e(A))^c$ and therefore

$$ (A^c)^l \subseteq (e(A))^c $$  \hspace{1cm} (2)
These two relations show that \((e(A))^c = (A^c)^l\) and the proof is completed. \(\Box\)

The following lemma is obvious from definition.

**Lemma 3.2.** Let \(A\) be a subset of \(X\). Then,

i) \(e(A \cap B) = e(A) \cap e(B)\);

ii) \(\bar{f}(e(A)) \subseteq e(f(A))\);

iii) \(\bar{f}^n = f^n\).

**Lemma 3.3.** Let \(f : X \to X\) be a continuous bijection and \(A\) be a subset of \(X\). Then \(\bar{f}(e(A)) = e(f(A))\).

**Proof.** According to previous Lemma \(\bar{f}(e(A)) \subseteq e(f(A))\). Conversely if \(K \in e(f(A))\), then \(f^{-1}(K) \subseteq f^{-1}(f(A))\). Also \(f^{-1}(f(A)) \subseteq A\) because \(f\) is a bijection. Therefore \(f^{-1}(K) \subseteq A\) and \(f(f^{-1}(K)) \in f(e(A))\). Also \(f^{-1}(f(K)) = K\). Hence \(K \in f(e(A))\) and therefore \(e(f(A)) \subseteq \bar{f}(e(A))\). \(\Box\)

**Theorem 3.1.** Let \(X\) be a compact metric space and \(f : X \to X\) be a continuous bijection. If \(f\) is chaotic in the sense of Block-Coppel’s, then so is \(\bar{f}\).

**Proof.** Let \(f\) be Block-Coppel’s chaotic, then there exist disjoint non-empty compact subsets \(J, K\) of \(X\) and a positive integer \(n\) such that \(J \cup K \subseteq f^n(J) \cap f^n(K)\). We claim that

\[ e(J) \cup e(K) \subseteq f^n(e(J)) \cap \bar{f}^n(e(K)). \]

Since \(J \subseteq f^n(J)\), then \(e(J) \subseteq e(f^n(J))\). According to Lemma 3.3 \(e(f^n(J)) = \bar{f}^n(e(J))\). Therefore \(e(J) \subseteq f^n(e(J))\). In a similar way

\[ e(J) \subseteq f^n(e(K)), e(K) \subseteq f^n(e(K)) \text{ and } e(K) \subseteq f^n(e(J)). \]

Therefore

\[ e(J) \cup e(K) \subseteq f^n(e(J)) \cap \bar{f}^n(e(K)). \]

On the other hand \(e(J) \cap e(K) = e(J \cap K) = \emptyset\) and according to Lemma 3.1 \(e(J), e(K)\) are compact, and the proof is completed. \(\Box\)

**References**

